Smooth, global PŁ functions are non-linear least squares

March 30, 2026 – Tufts Math

Chris Criscitiello

with

Nicolas Boumal, Quentin Rebjock

UPenn

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Race to the bottom: https://www.racetothebottom.xyz/posts/global-polyak-lojasiewicz/

UPenn

PŁ condition

$$\min_{x \in \mathbb{R}^n} f(x)$$

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Function classes:

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- Smooth, μ-strongly convex functions
- Functions with L-Lipschitz gradient
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- **This talk**: smooth, globally PŁ functions

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μ-globally PŁ: $\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}^n, \quad \mu > 0$

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Polyak '63, Łojasiewicz ’65, Karimi et al. ‘16, Liu et al. ’20

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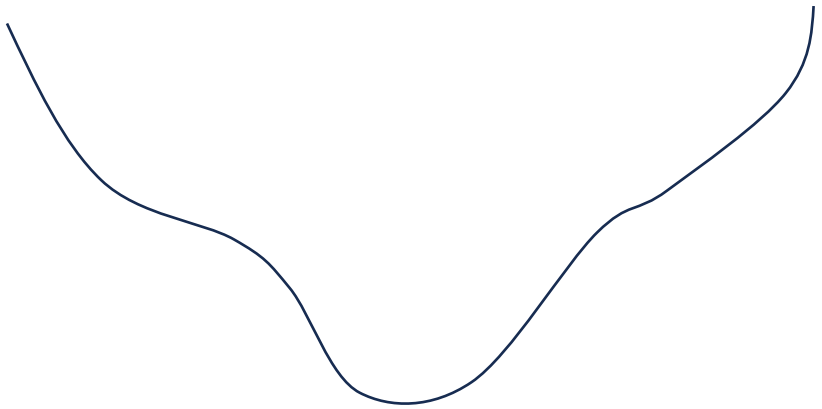
All critical points are global mins $\nabla f(x) = 0 \implies f(x) = f^*$



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All critical points are global mins $\nabla f(x) = 0 \implies f(x) = f^*$

μ-strongly convex functions are μ-globally PL.

But the converse is not true: f need not be convex!

Why care about PŁ?

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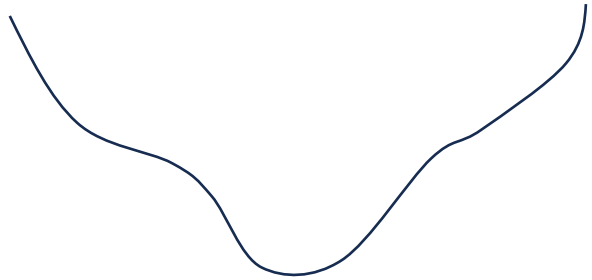

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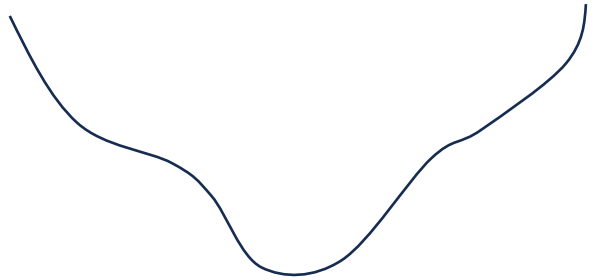

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Gradient descent/flow behaves nicely

L-Lipschitz gradient and μ-PŁ imply linear convergence (short proof!)

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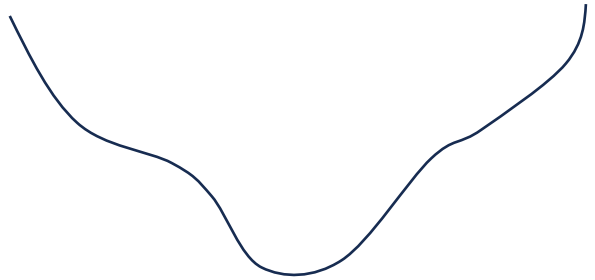

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$$f(x_{k+1}) - f^* \leq \left(1 - \frac{\mu}{L}\right)(f(x_k) - f^*) \qquad x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$$

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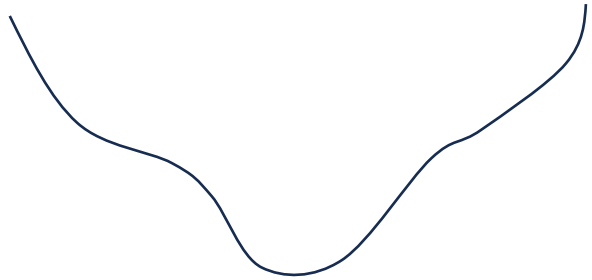

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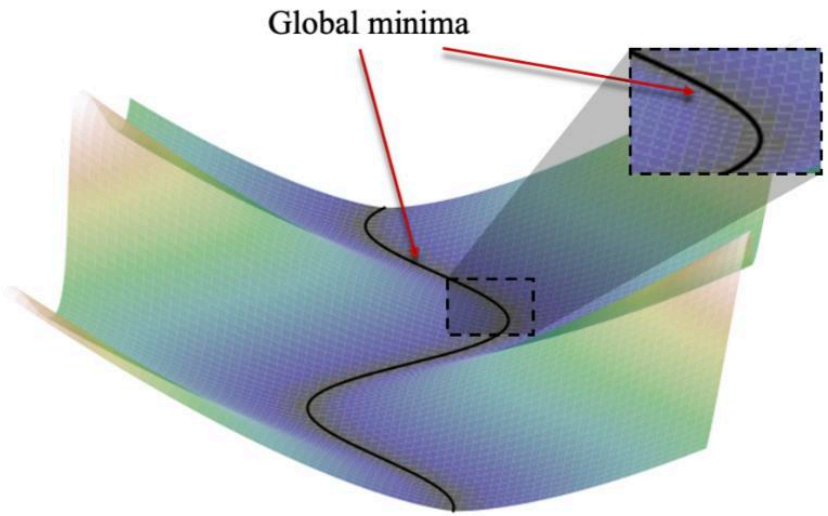

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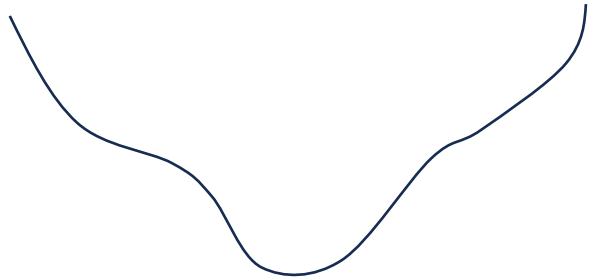

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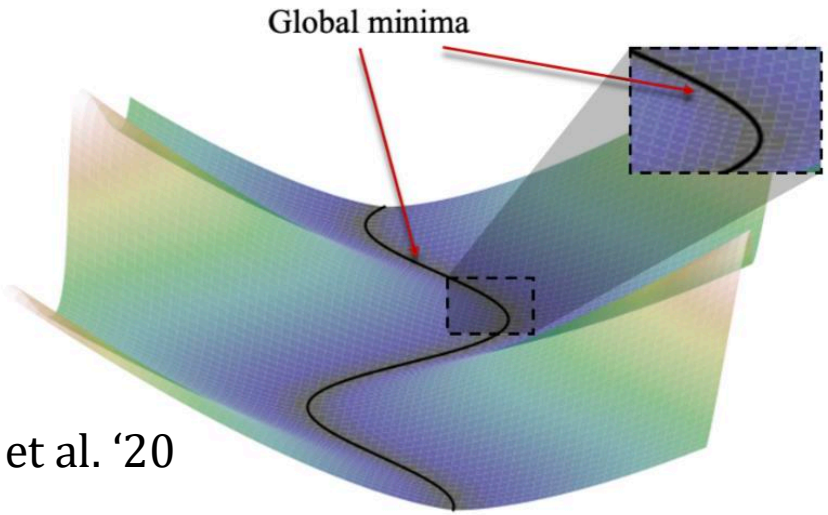

Liu et al. ‘20

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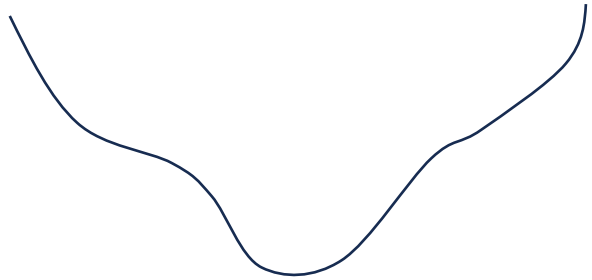

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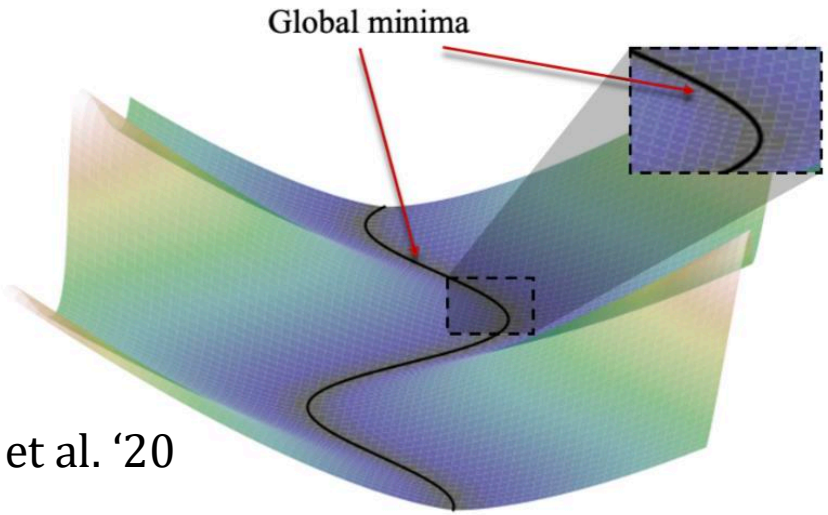
Overparameterization!

Applications: Deep learning (Liu et al. '20),
control (Fazel et al. '18),
statistics (Chewi et al. '20)



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Example

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \qquad \forall x \in \mathbb{R}^n$$

Example: non-linear least squares

$$f(x) = \|F(x) - b\|^2 \qquad F: \mathbb{R}^n \to \mathbb{R}^k$$

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$$f(x) - f^* \leq \frac{1}{2\sigma(x)^2}\|\nabla f(x)\|^2 \qquad \sigma(x) = \sigma_k(\mathrm{D}F(x)^*)$$

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We show all globally PŁ functions are of this form!

What do PŁ functions look like?

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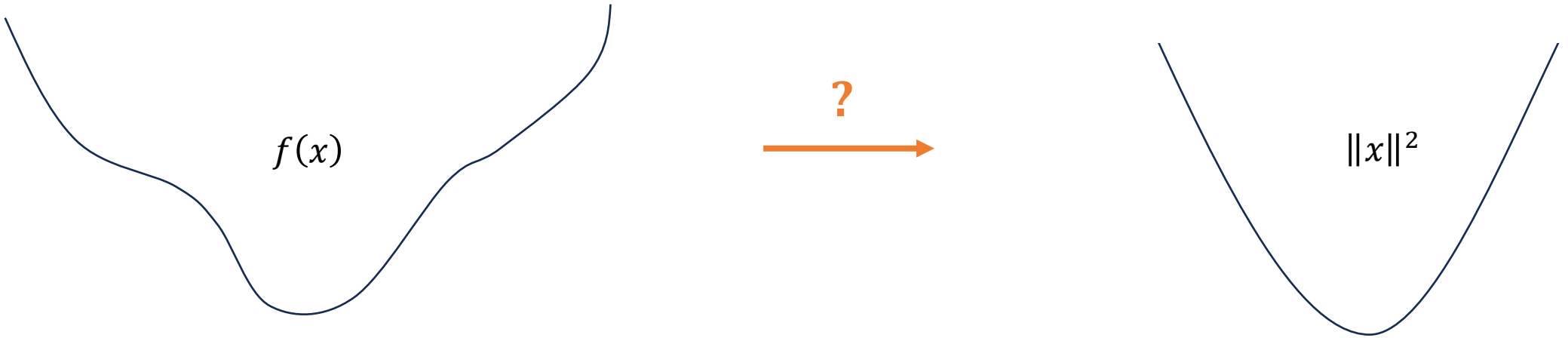

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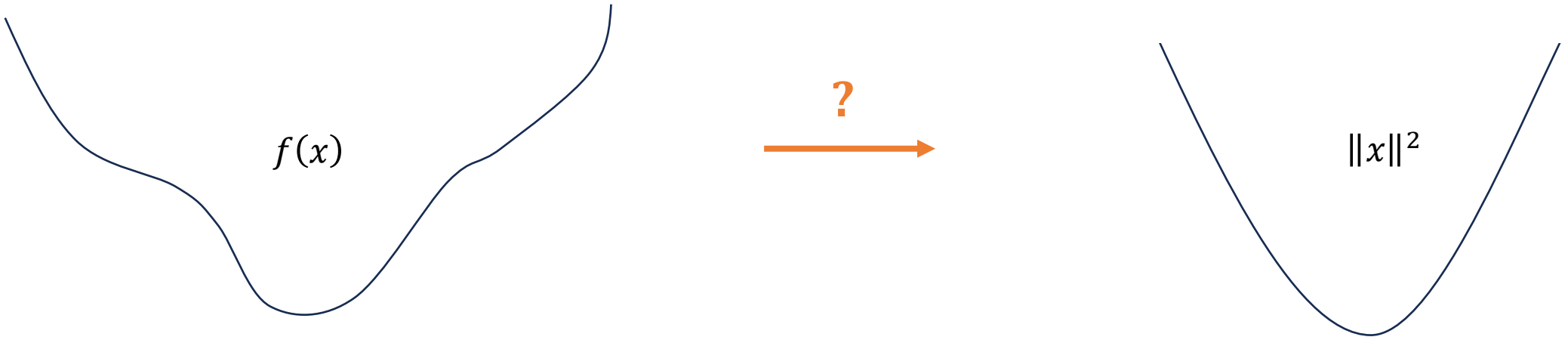

If f is smooth and globally PŁ with unique minimizer, does there exist a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) = f^* + \|\phi(x)\|^2 \ ?$$

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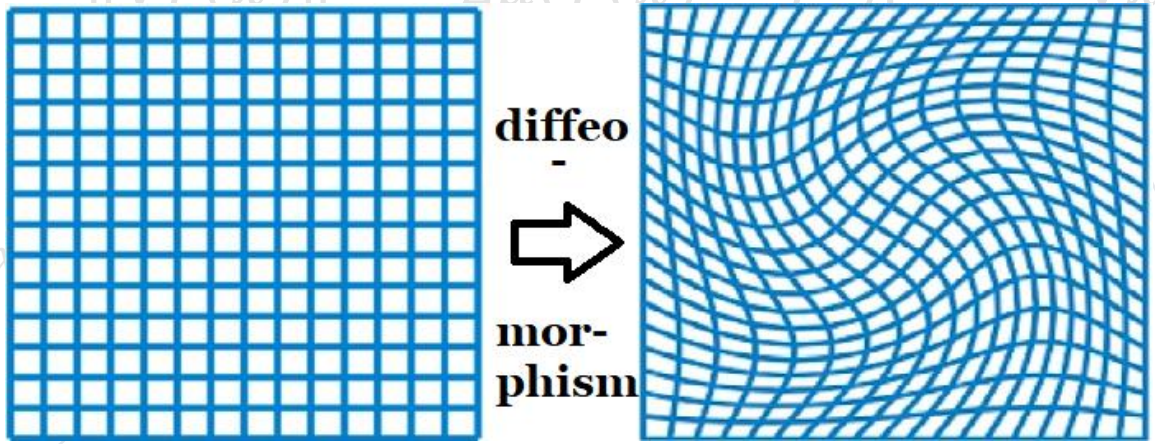

A diffeomorphism is a smooth map $\mathbb{R}^n \to \mathbb{R}^n$ with smooth inverse

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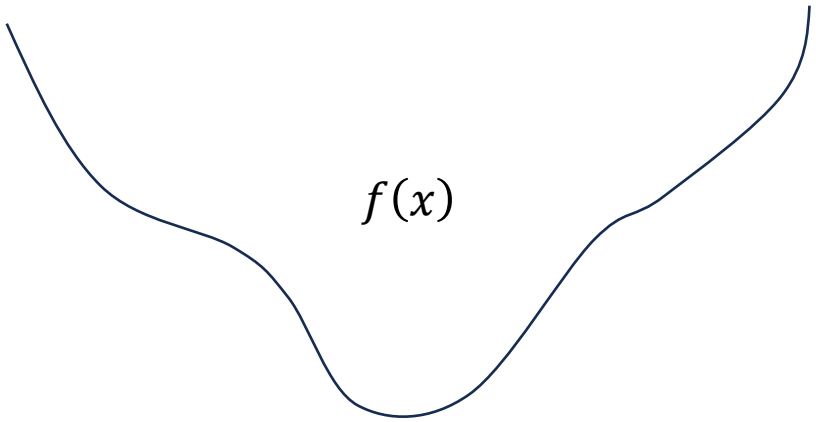

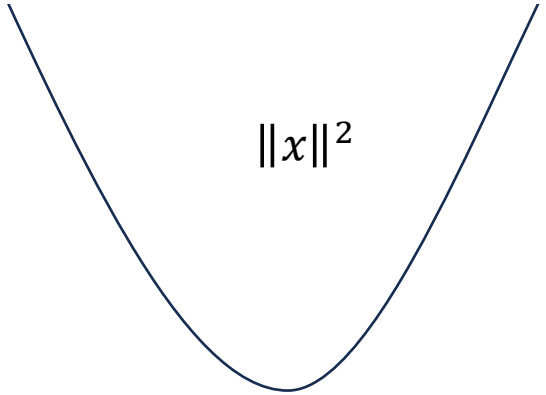

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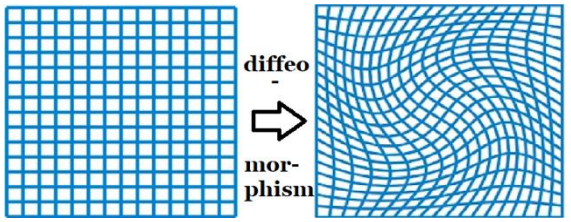

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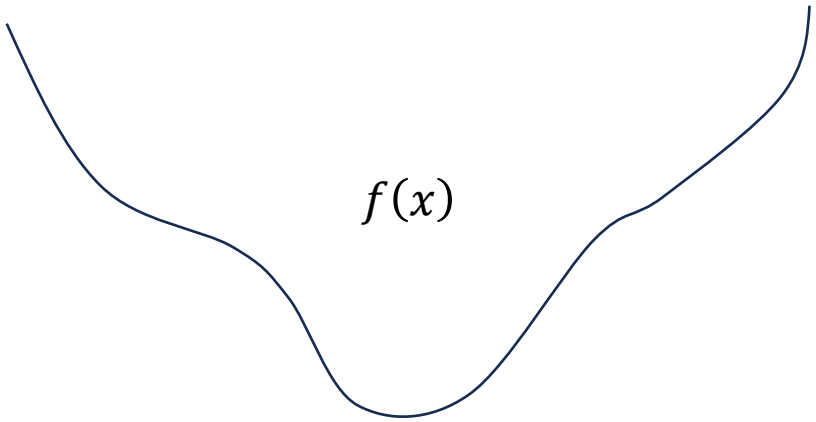

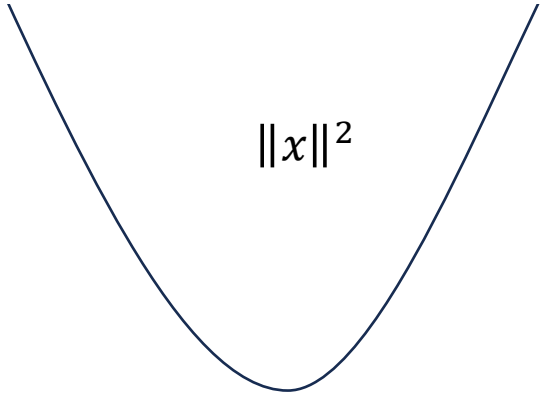

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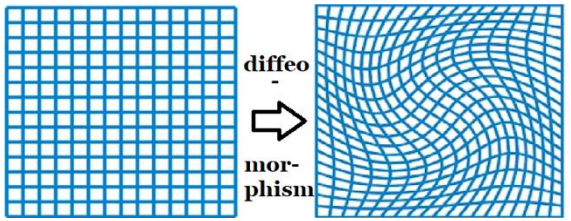

Baby Thm: **Yes**

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Cor: There is a complete flat Riemannian metric on $\mathbb{R}^n$ under which f is geodesically convex.

(compare with Rapcsák & Csendes '93)

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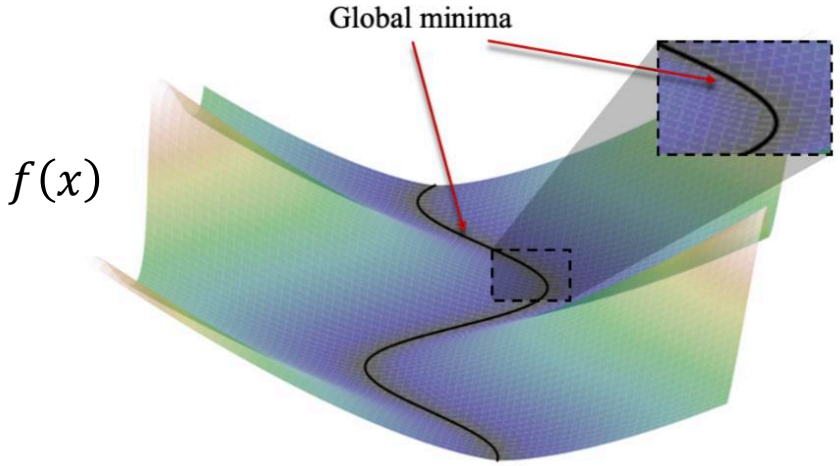

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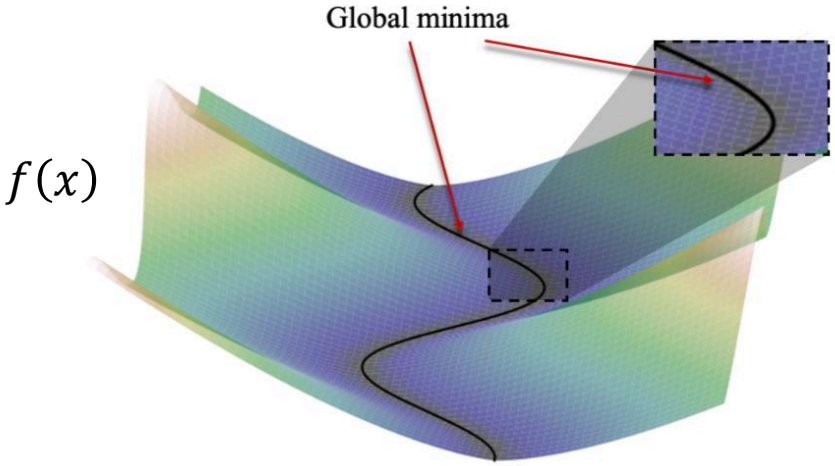

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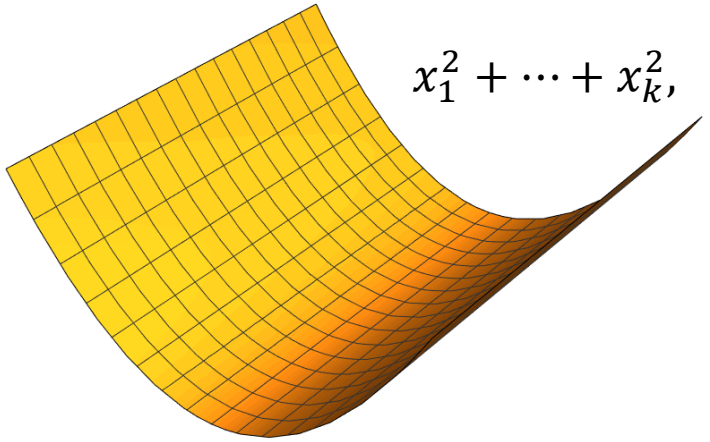

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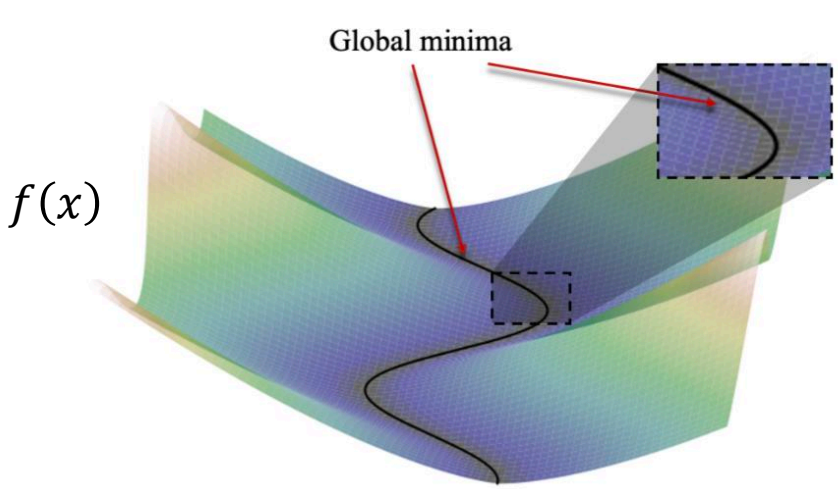

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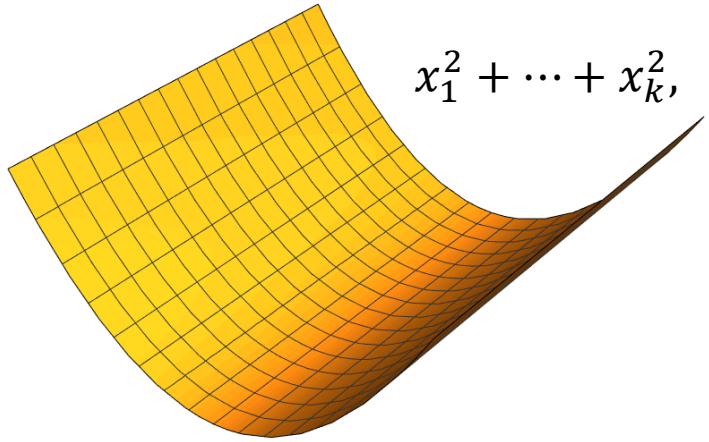

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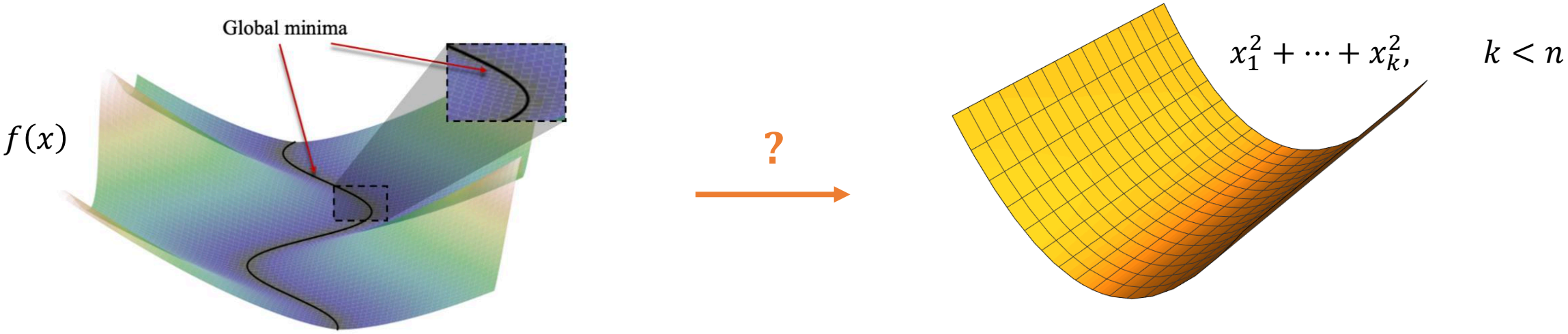

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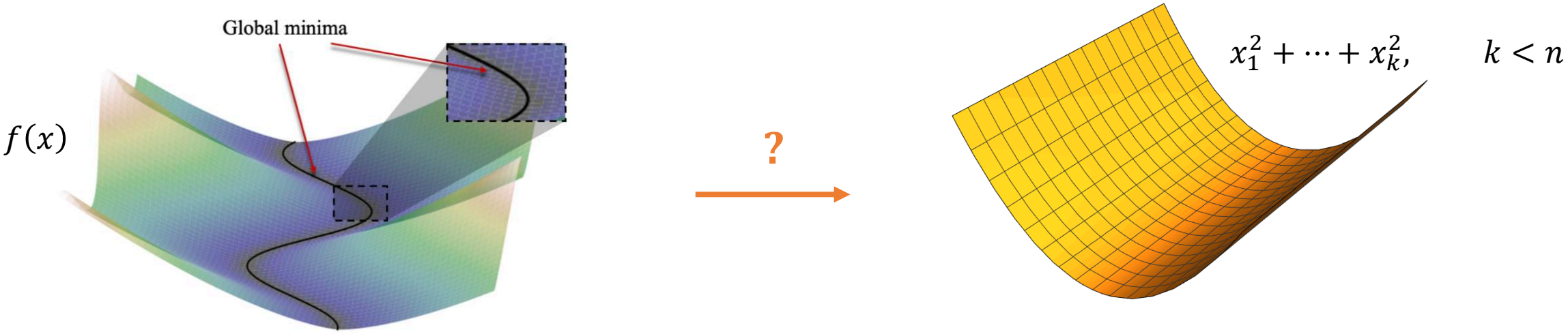

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Short answer: No in general. Yes if $S \cong \mathbb{R}^{n-k}$.

Main results

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S cannot be a circle, sphere, cylinder, …

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Implies $\phi(S) = S \times \{0\}$. Converse:

Thm (Building PŁ fcts): Let S be a submanifold of $\mathbb{R}^n$. If there exists a diffeomorphism

$$\phi: \mathbb{R}^n \to S \times \mathbb{R}^k \text{ with } \phi(S) = S \times \{0\}$$

then S is the argmin set of a globally PŁ function.

What can S be?

***Cor** (Characterization of S)*: Let S be a smooth manifold (without boundary).

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***Cor** (Characterization of S)*: Let S be a smooth manifold (without boundary).

The following are equivalent:

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Proof also relies on a deep result of McMillan ‘61, Stallings ‘62, Luft ‘87, Perelman ‘03.

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***Cor** (Characterization of S)*: Let S be a smooth manifold (without boundary).

The following are equivalent:

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What are all contractible smooth manifolds (without boundary)?

Certainly includes $\mathbb{R}^m$. Any others?

Doesn't include circle, sphere, closed disk, cross, …

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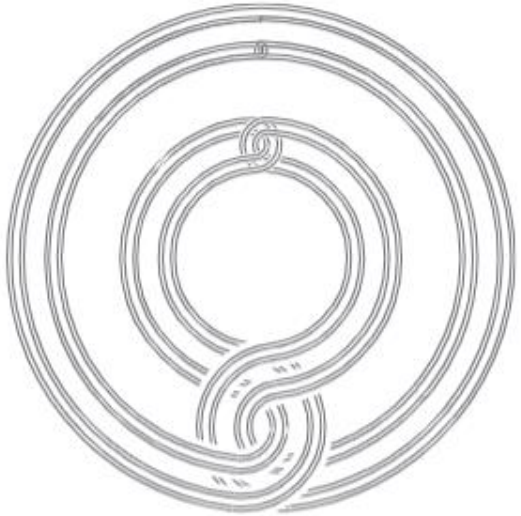
These manifolds are crazy!



Famous Whitehead manifold

Calegari '19

What if S isn't crazy?

Cor: If S is diffeomorphic to $\mathbb{R}^m$ (e.g., if $\dim S \leq 2$), then there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

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Summary:

Either f is simple, i.e., $f\big(\phi^{-1}(x)\big) = x_1^2 + \cdots + x_k^2$

Or S has to be wild, i.e., smooth and contractible but not diffeo to $\mathbb{R}^m$

Remainder of talk

Baby Thm: If f is smooth and globally PŁ with unique minimizer, there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

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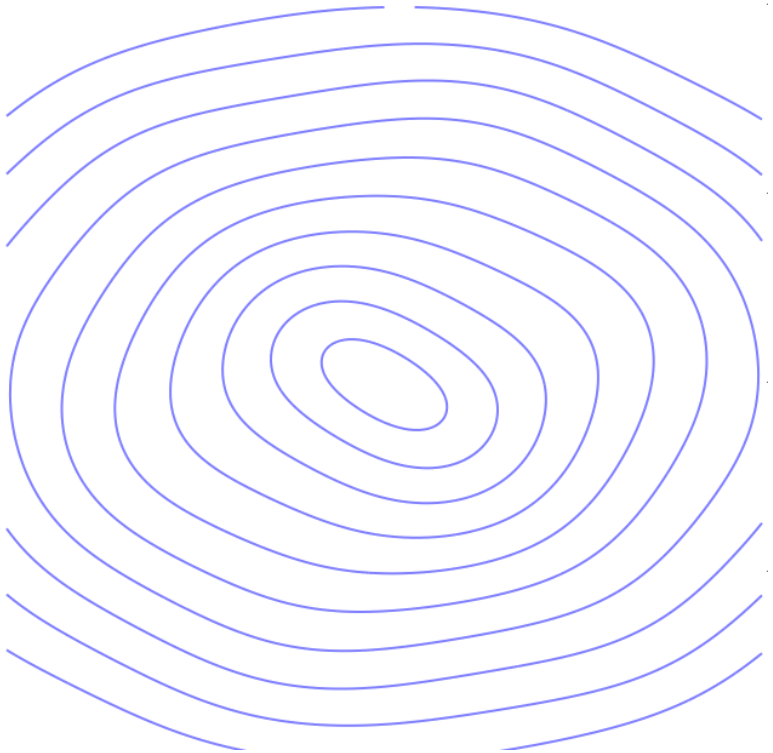

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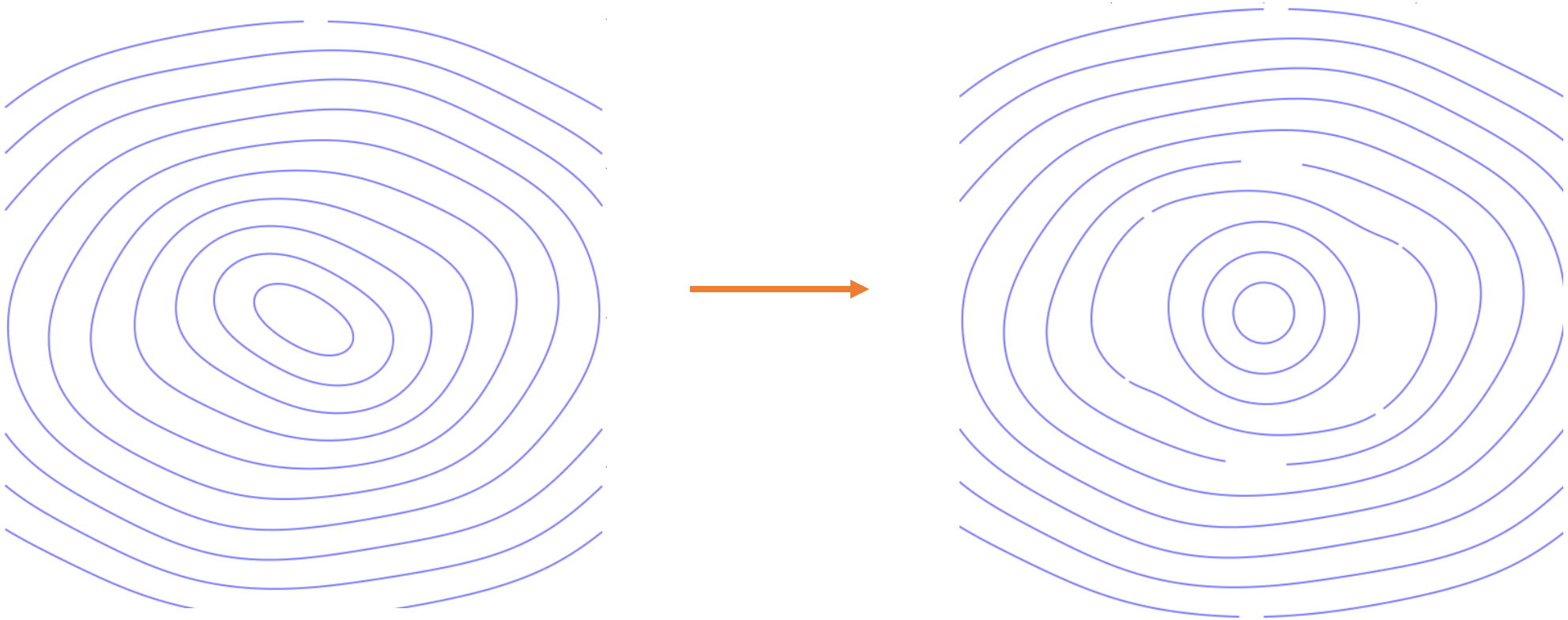

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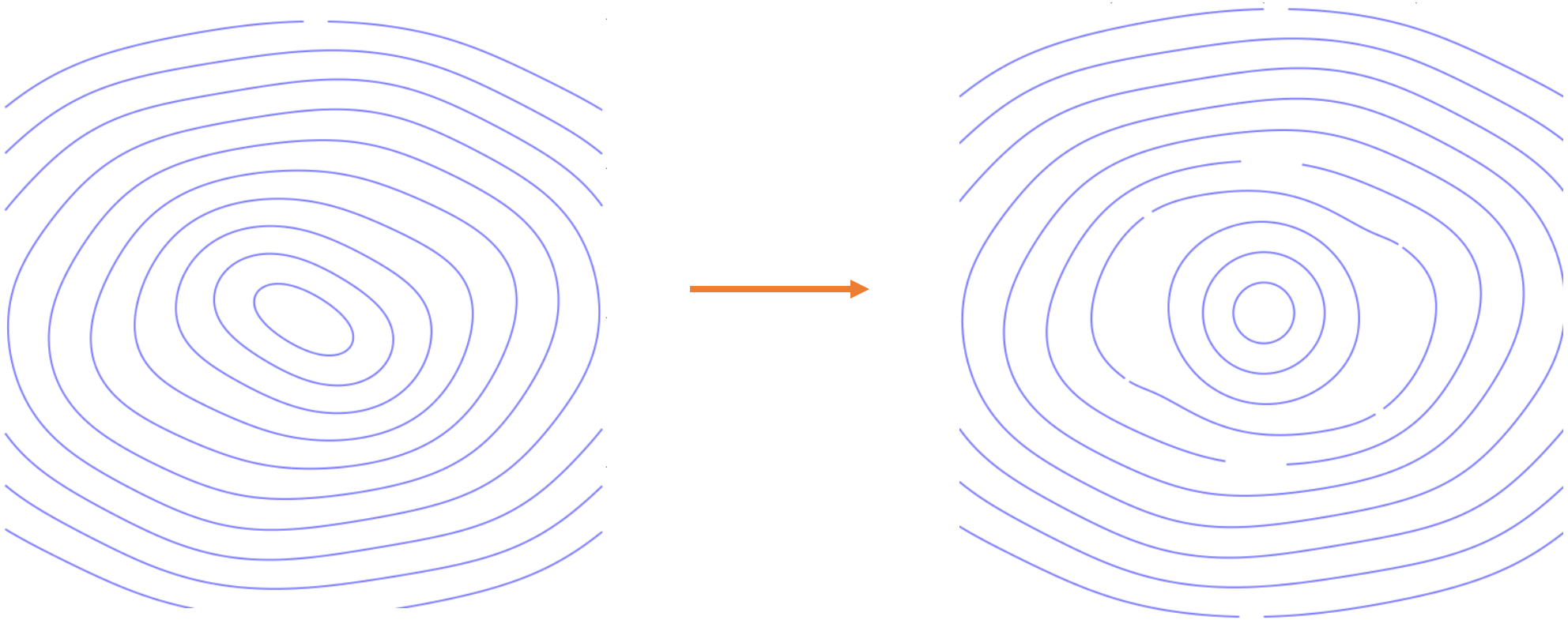

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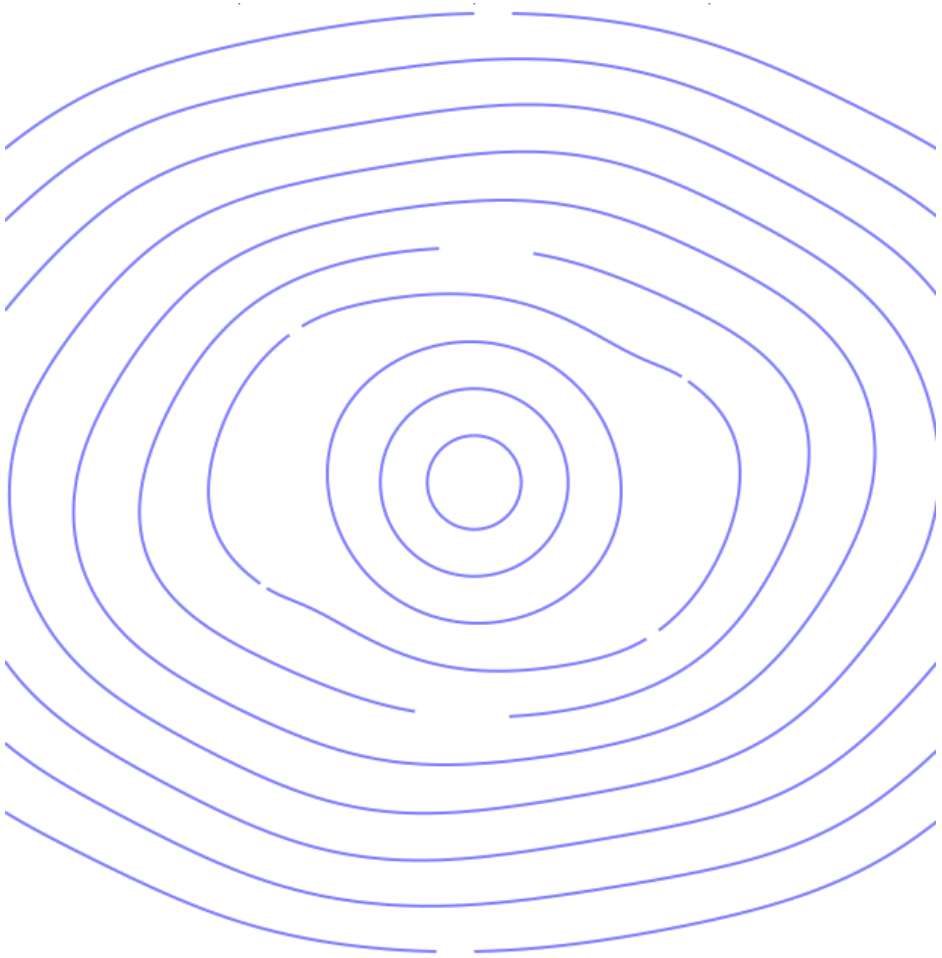

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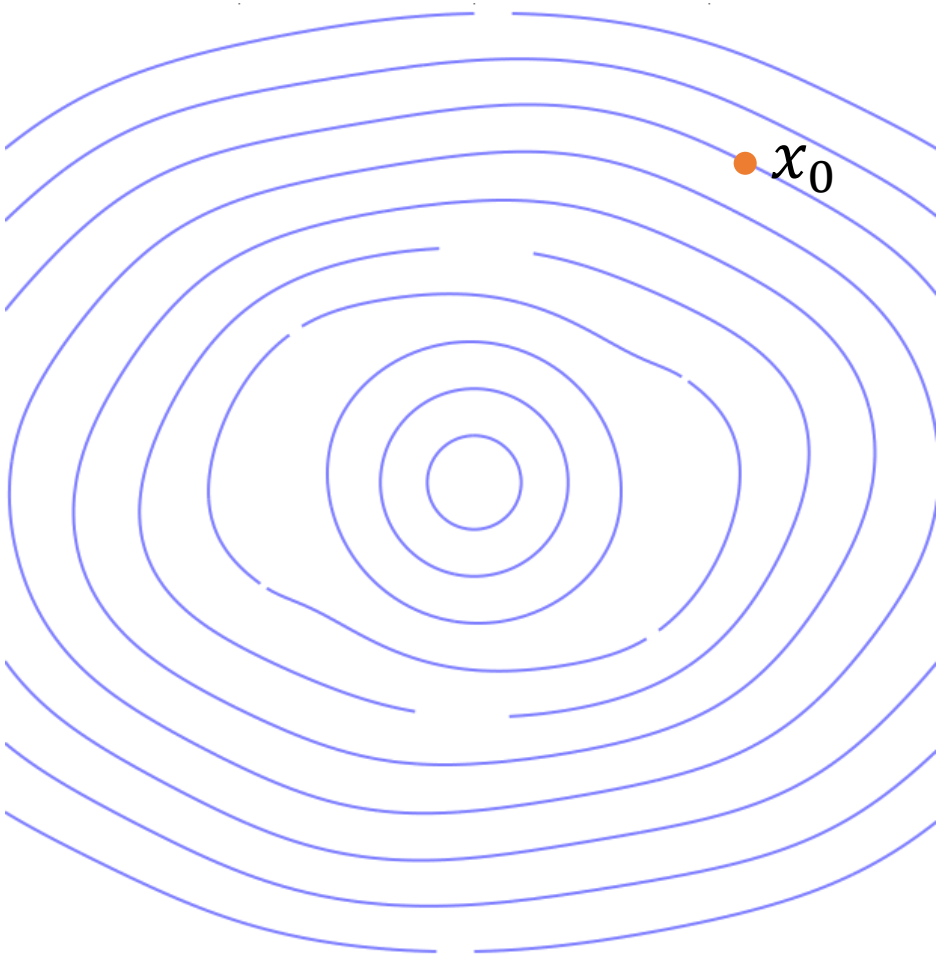

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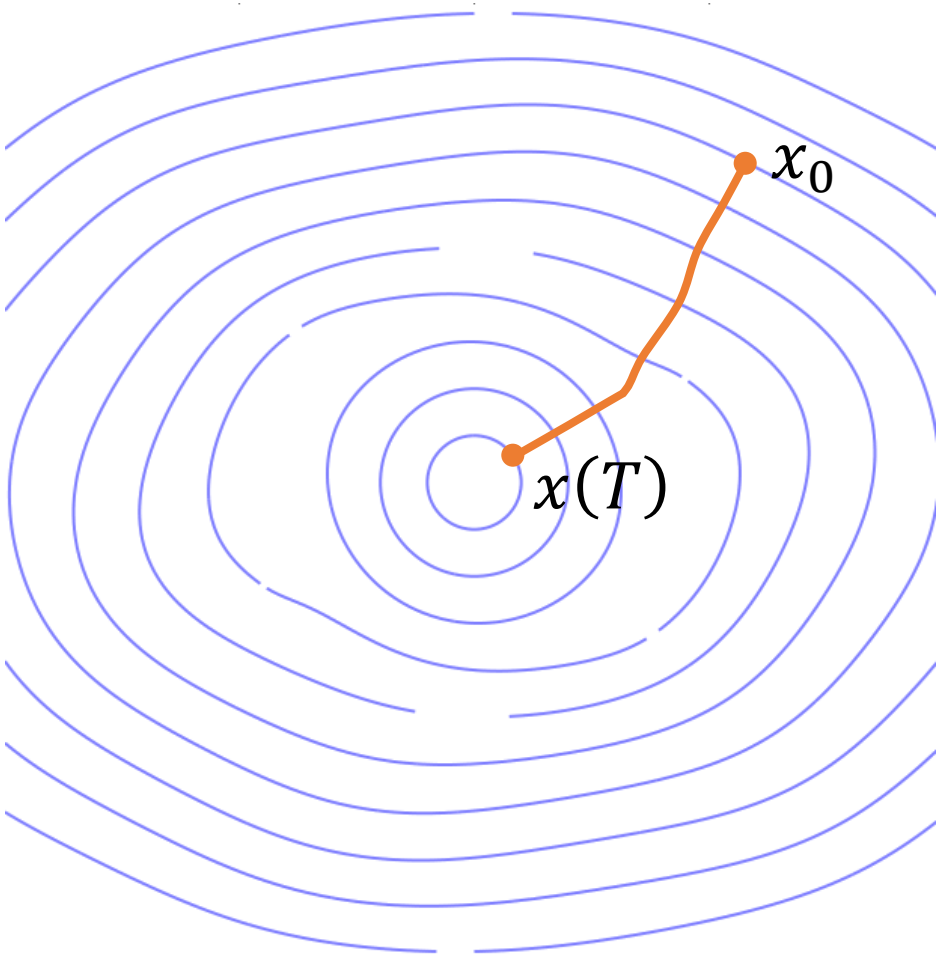

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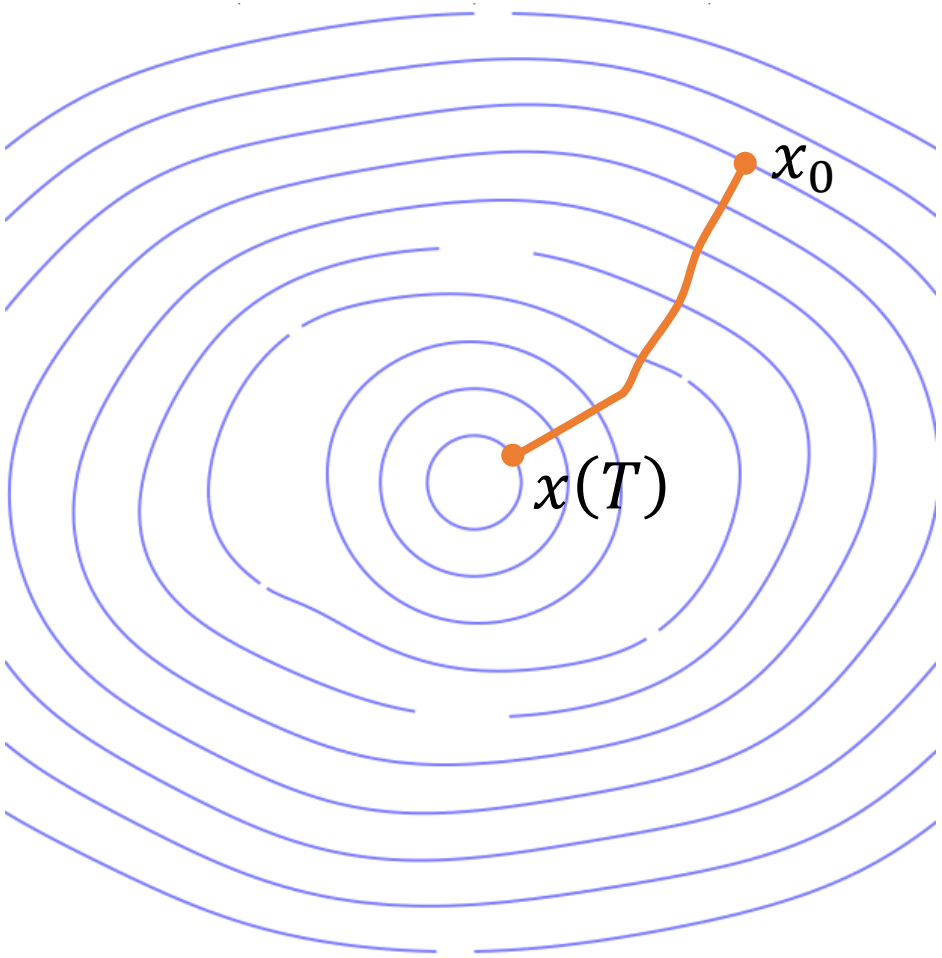

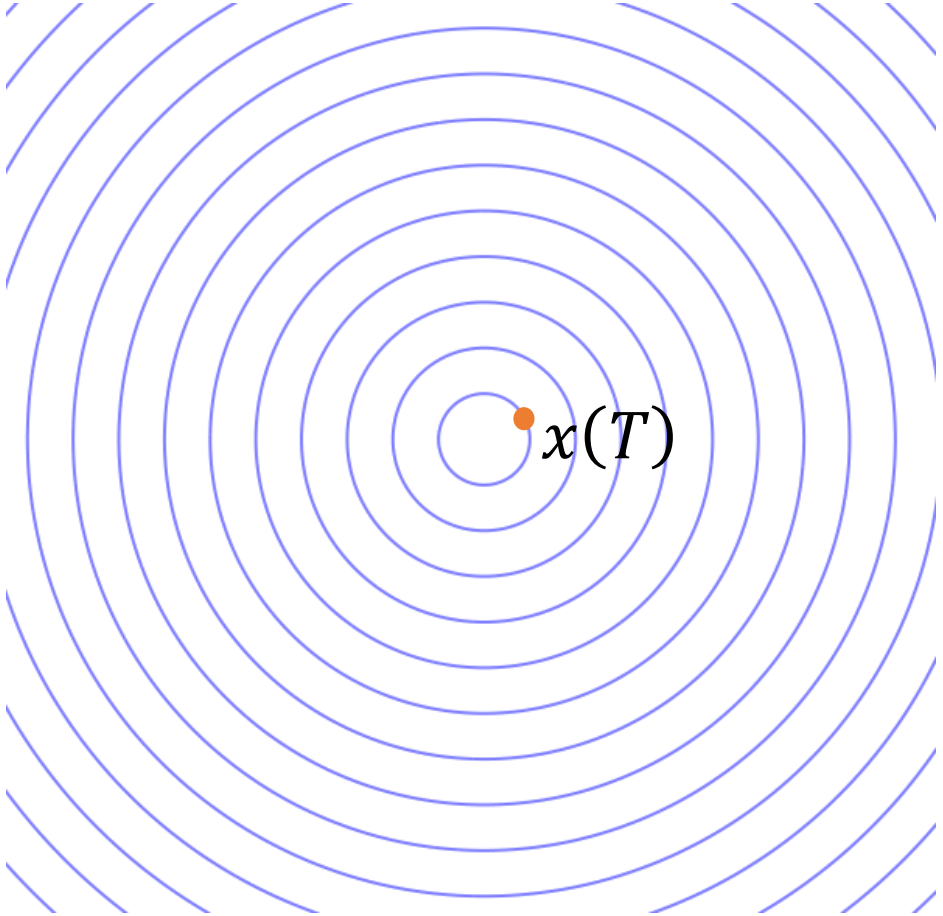

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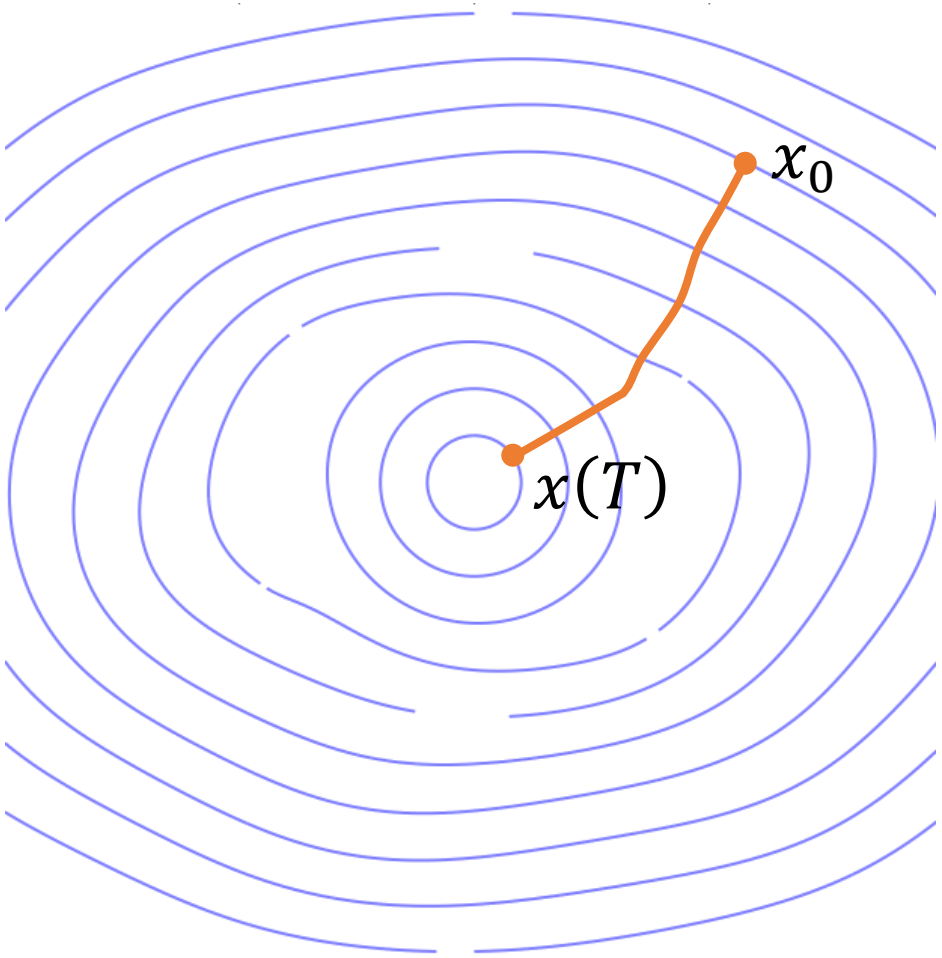

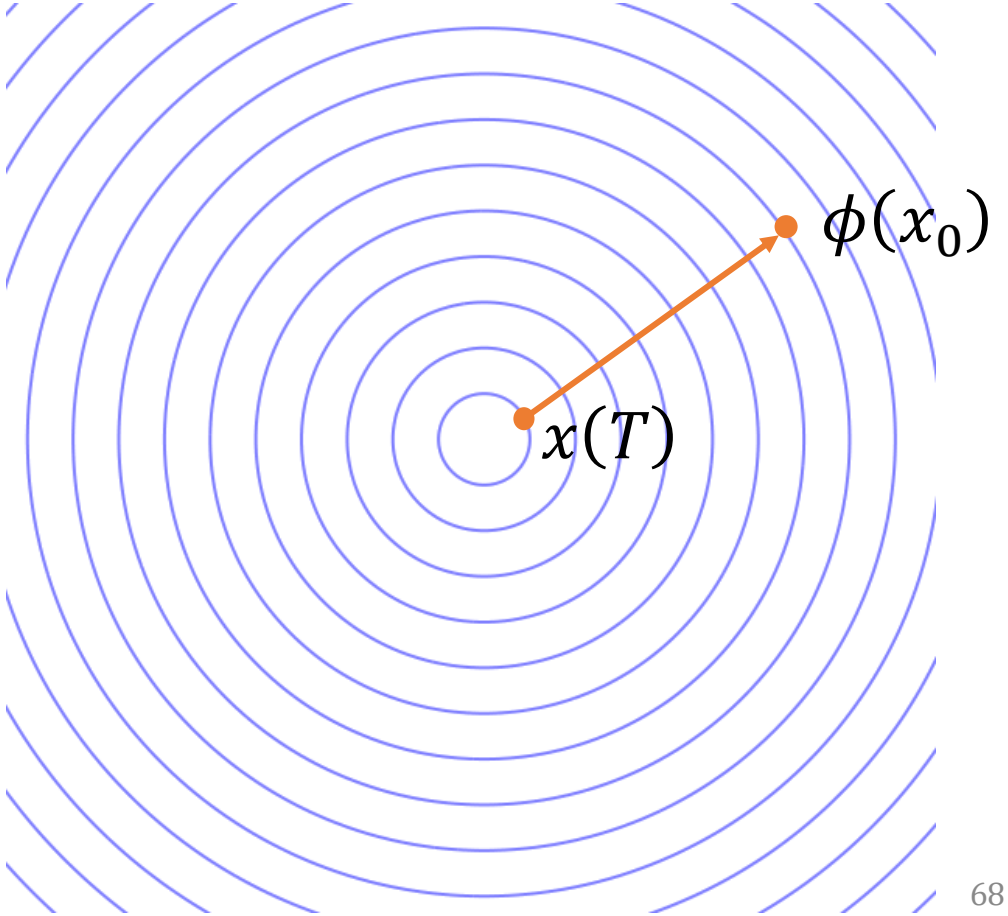

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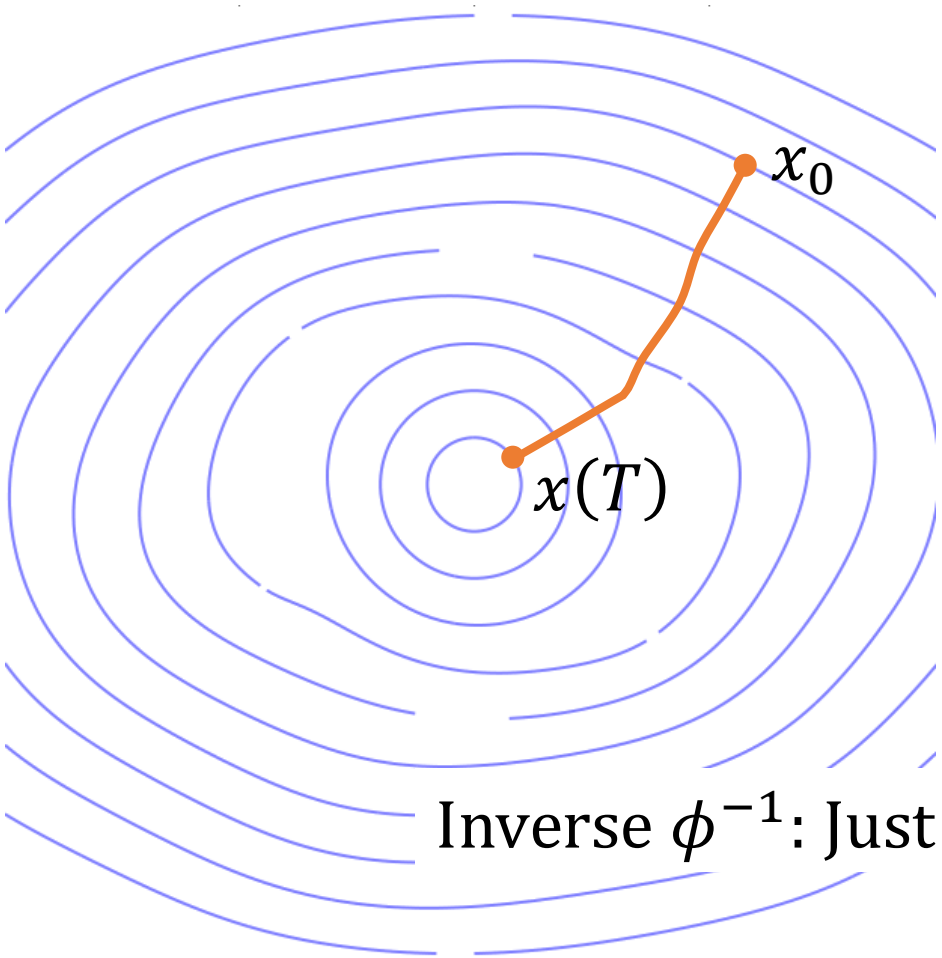

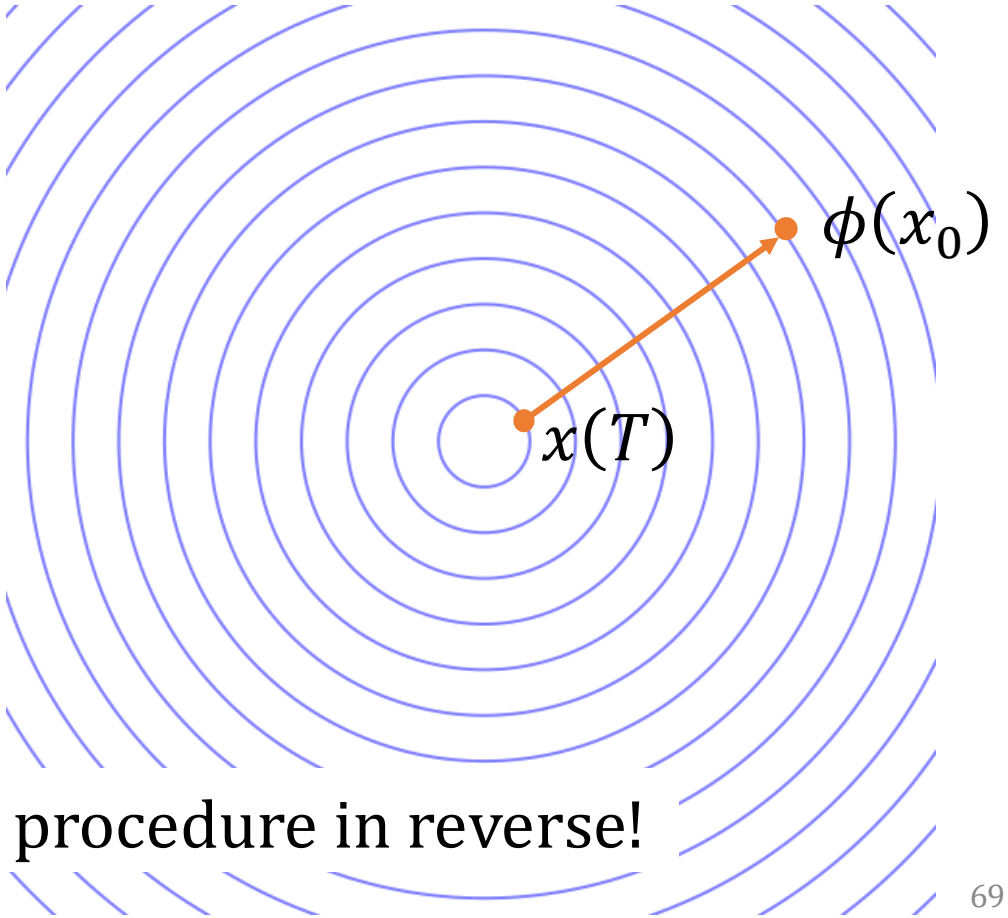

Inverse ϕ^{-1}: Just run the procedure in reverse!

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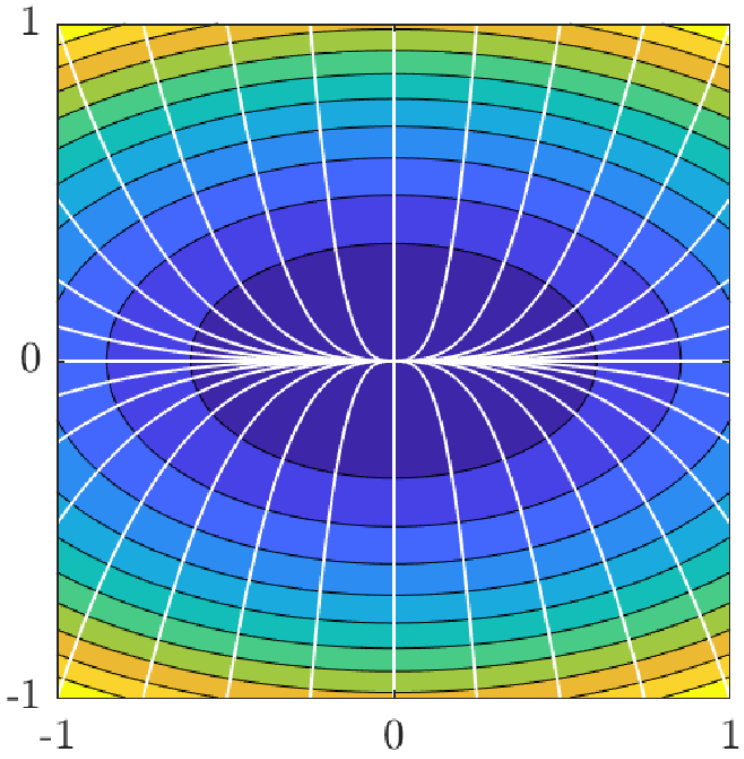

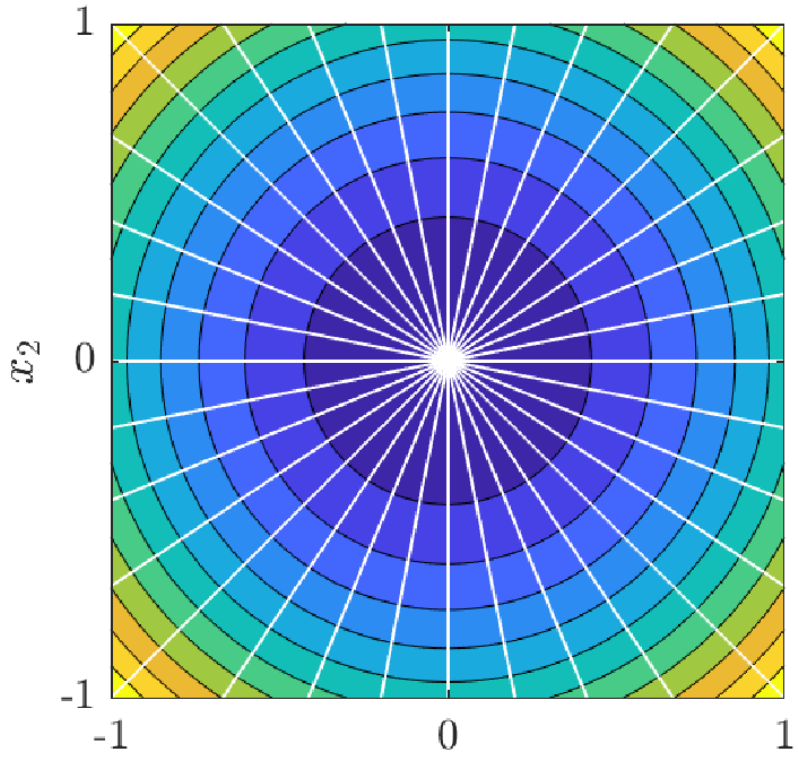

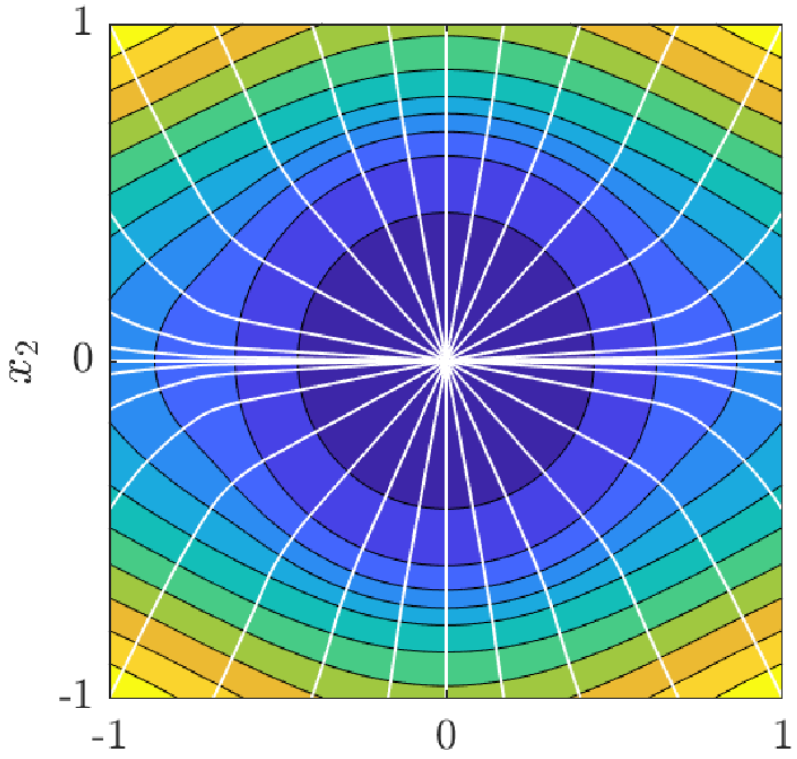

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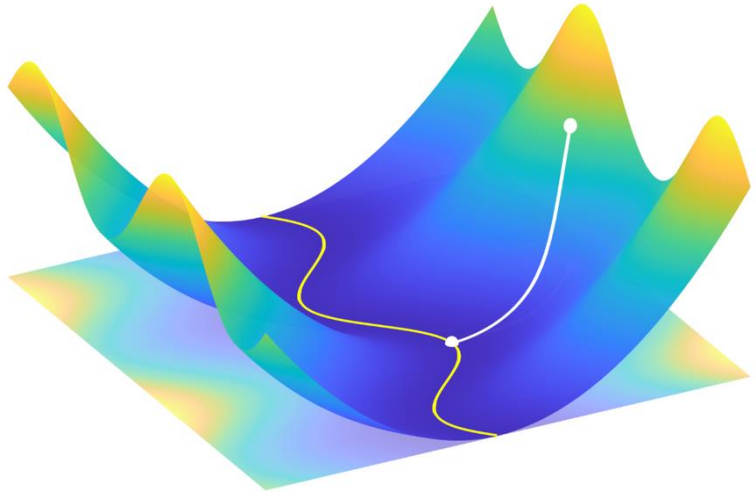

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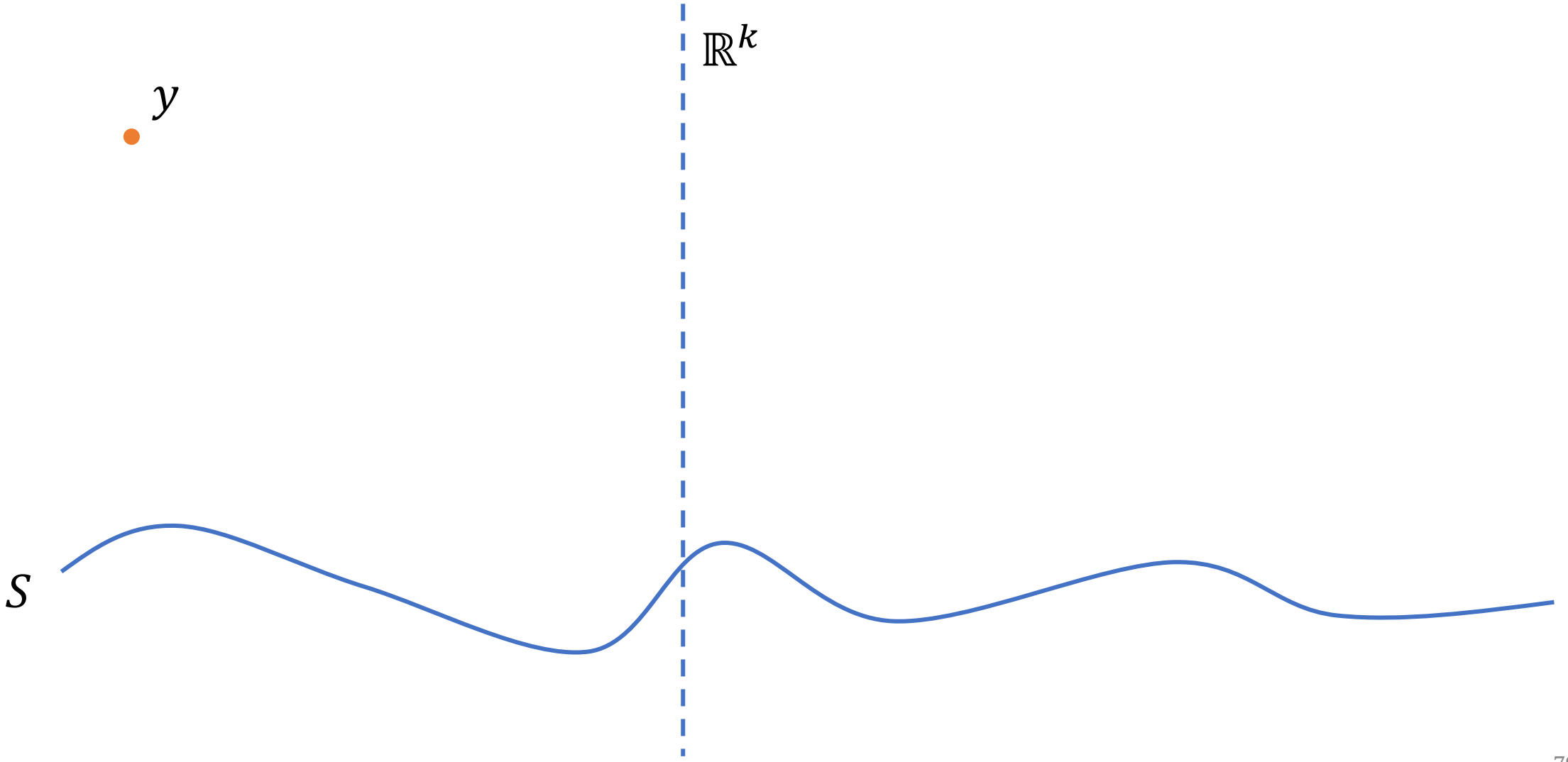

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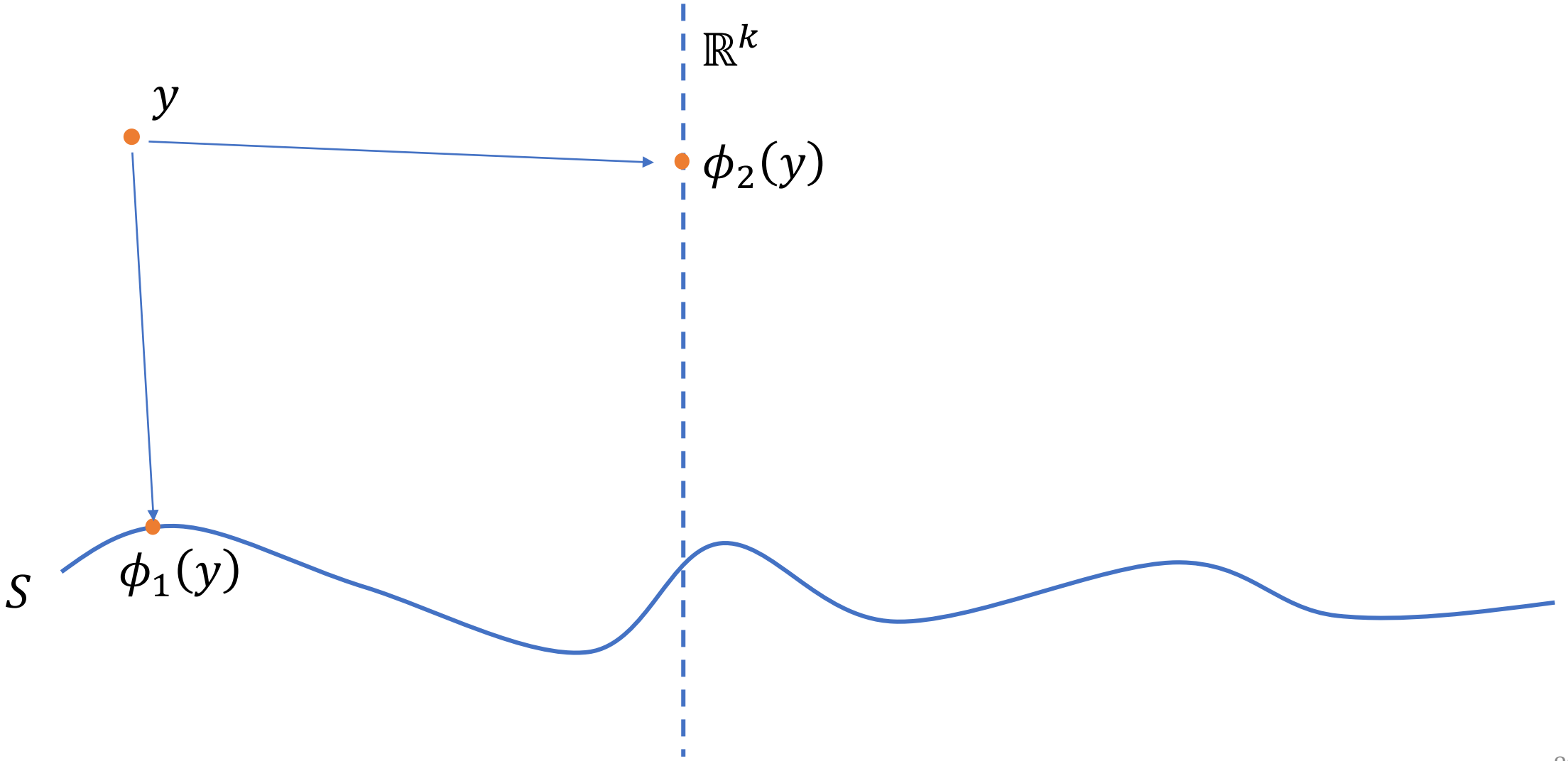

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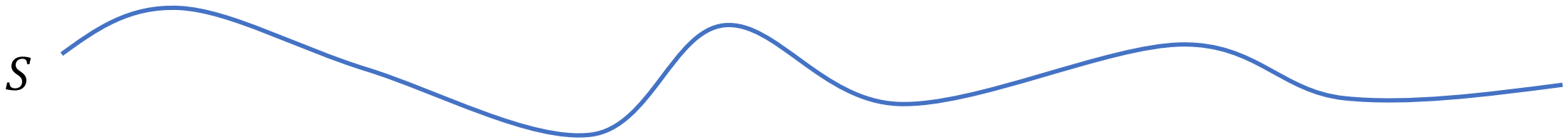

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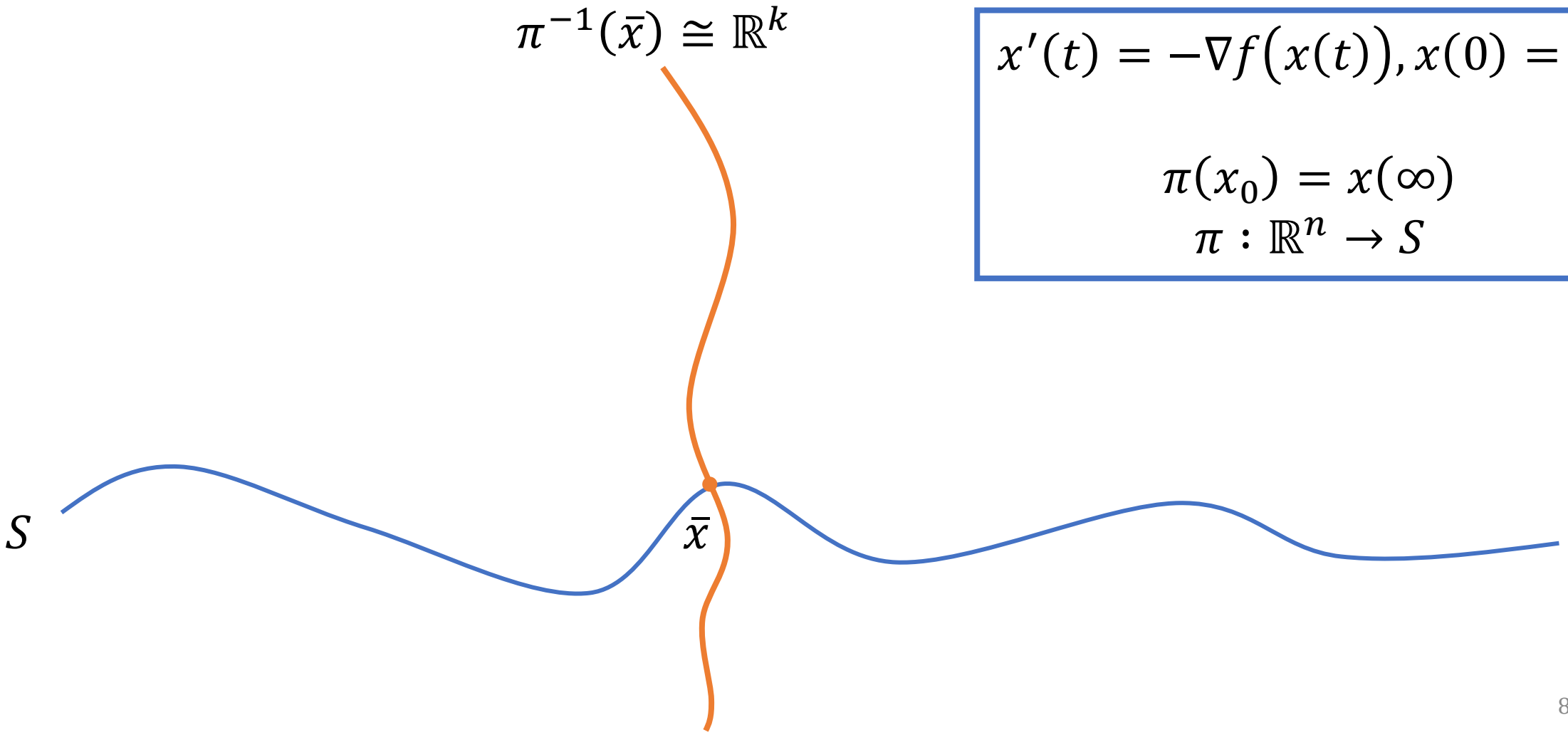

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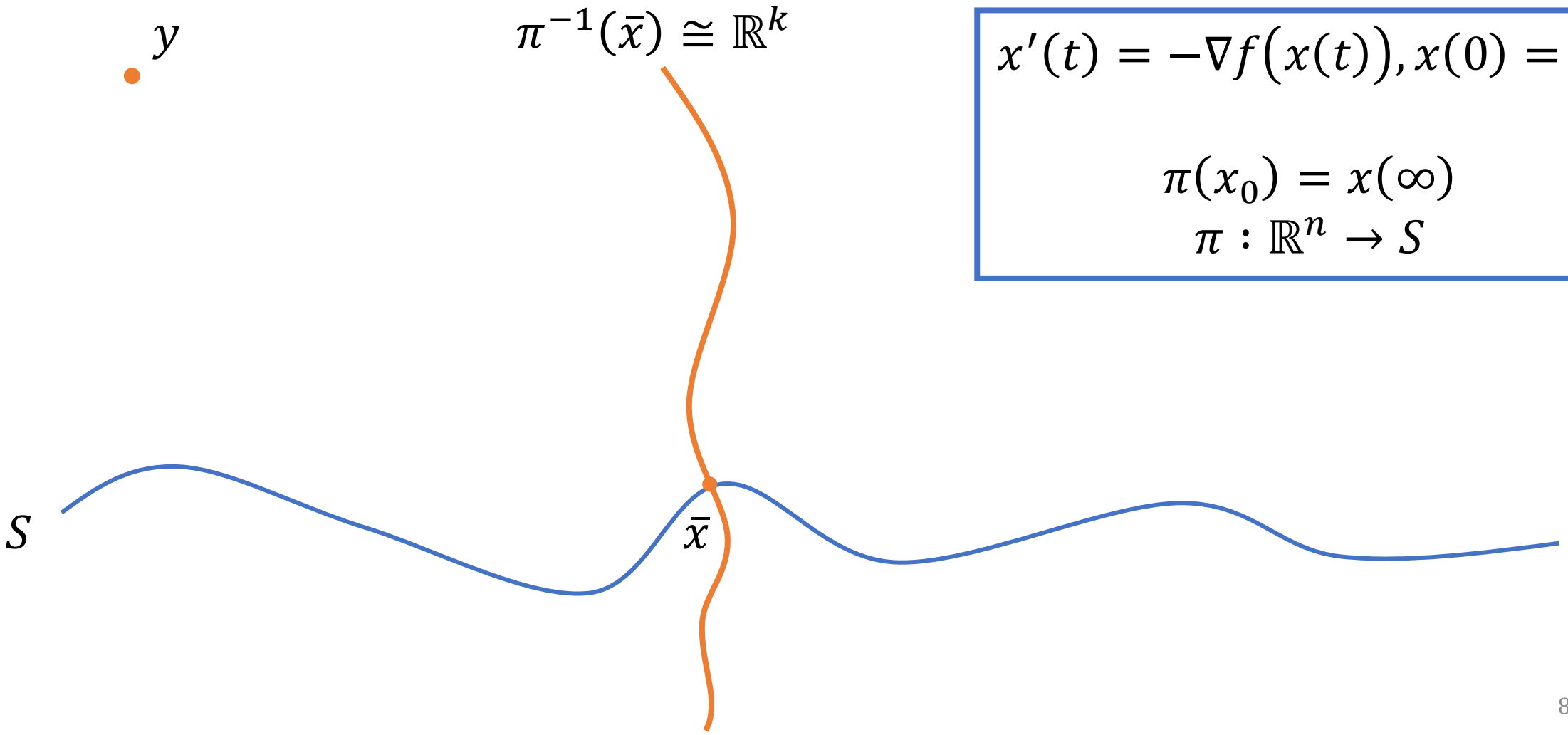

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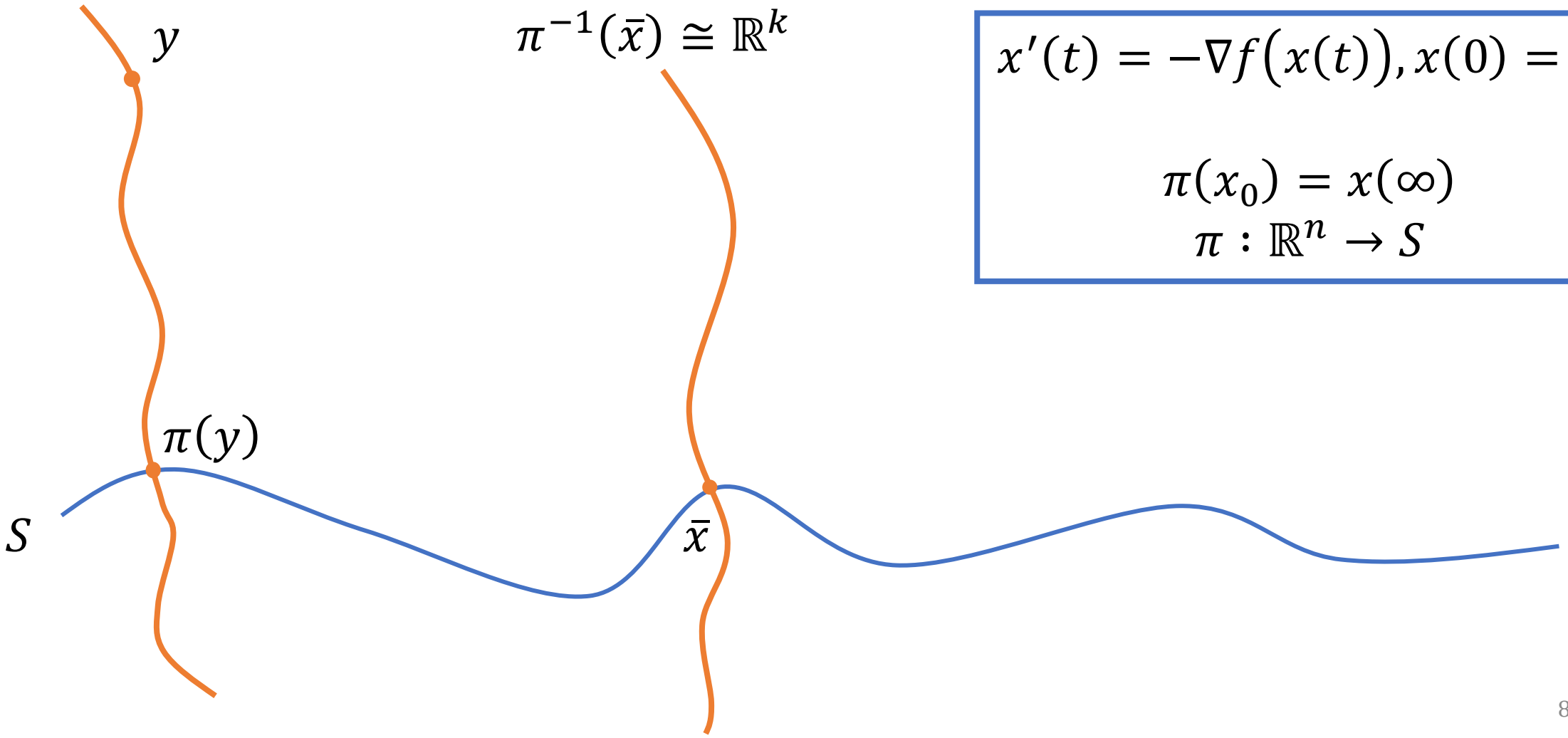

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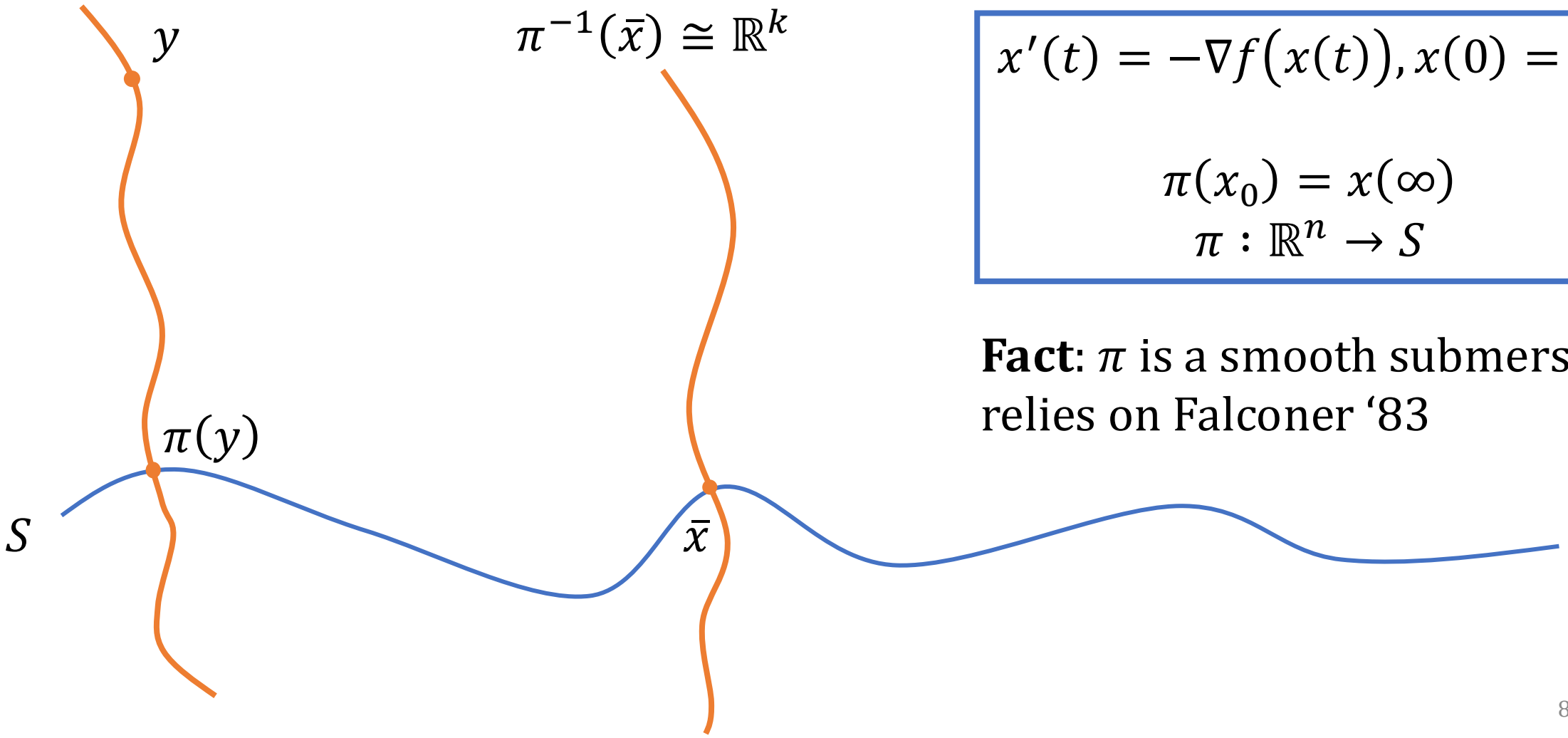

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Fact: π is a smooth submersion, relies on Falconer '83

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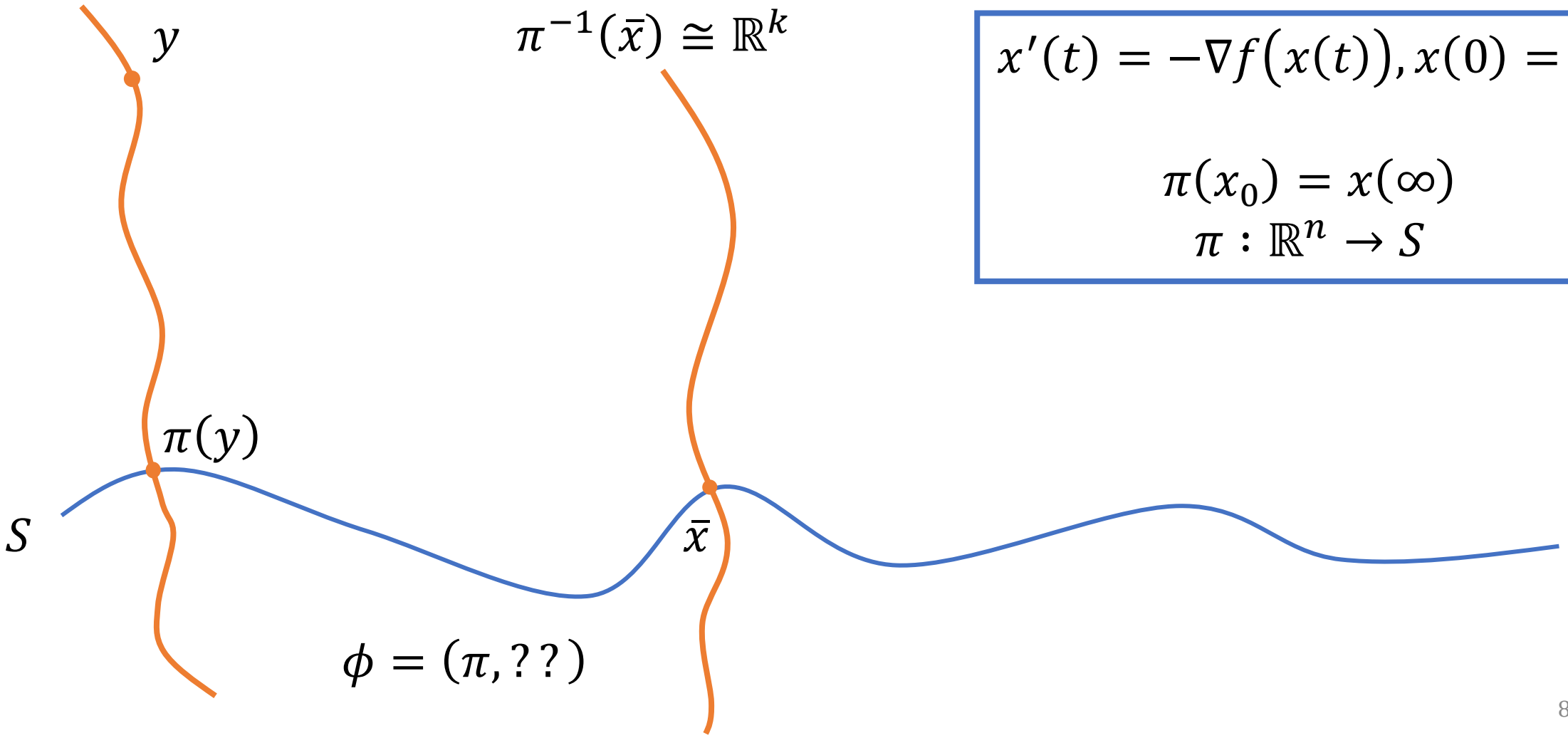

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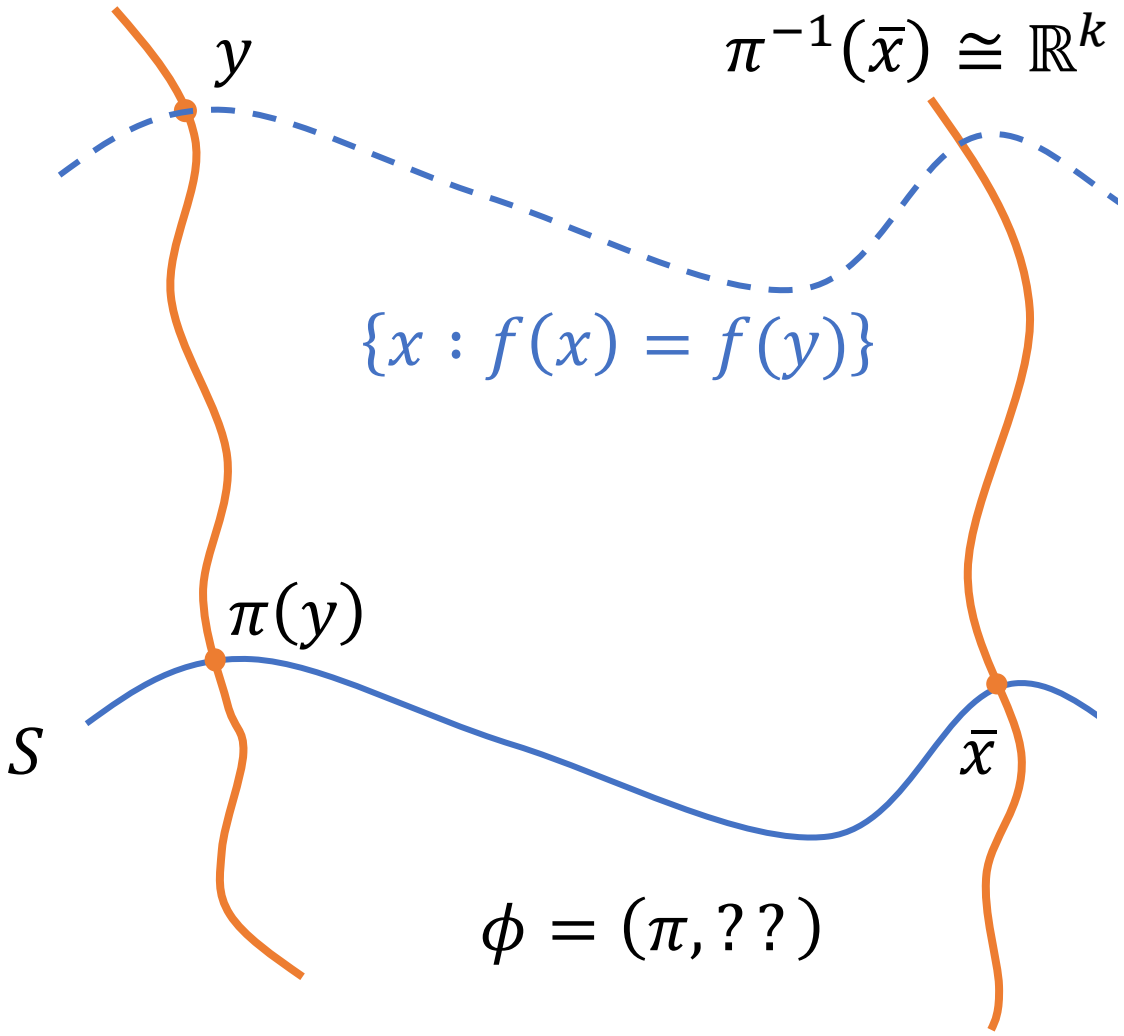

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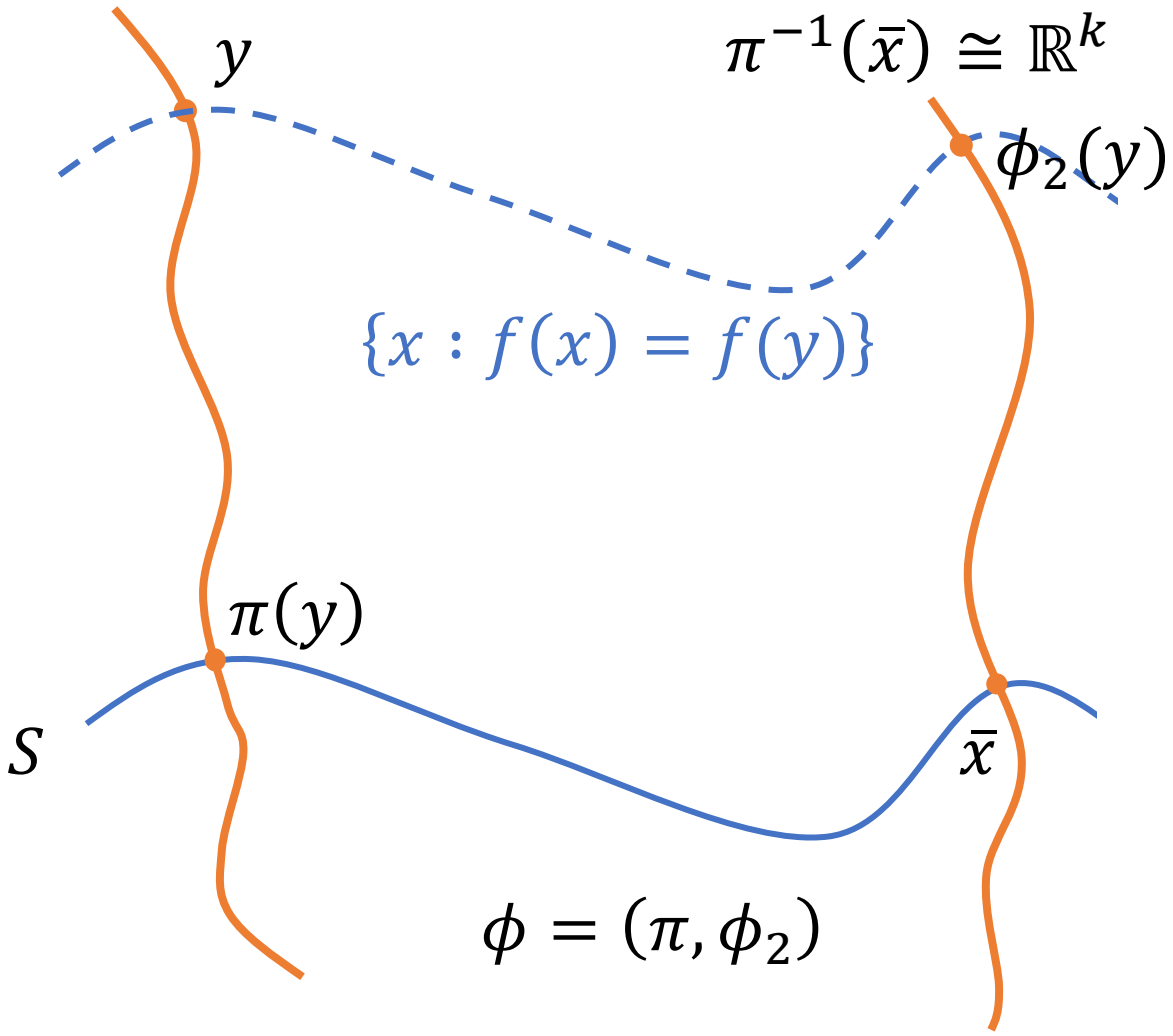

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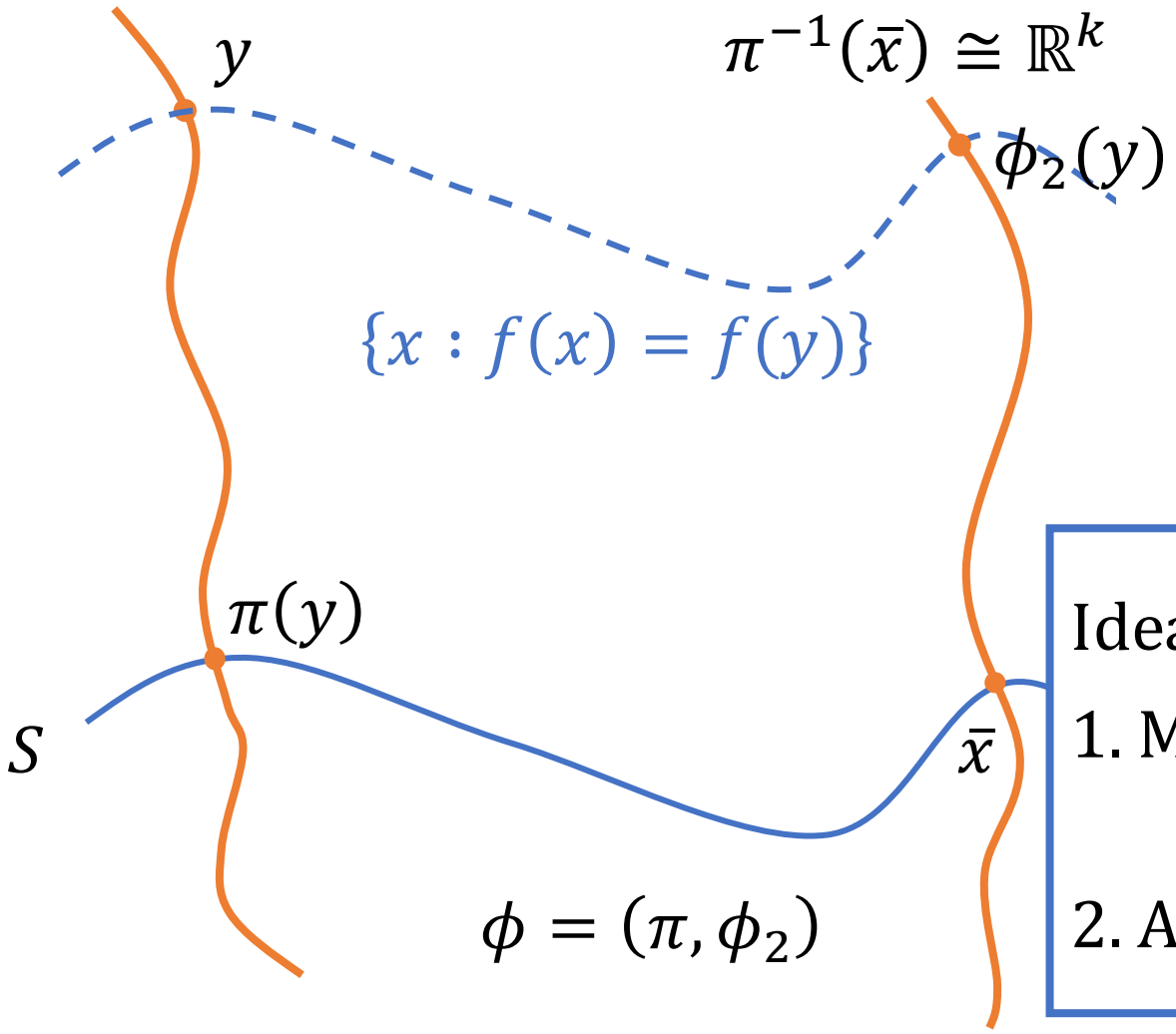

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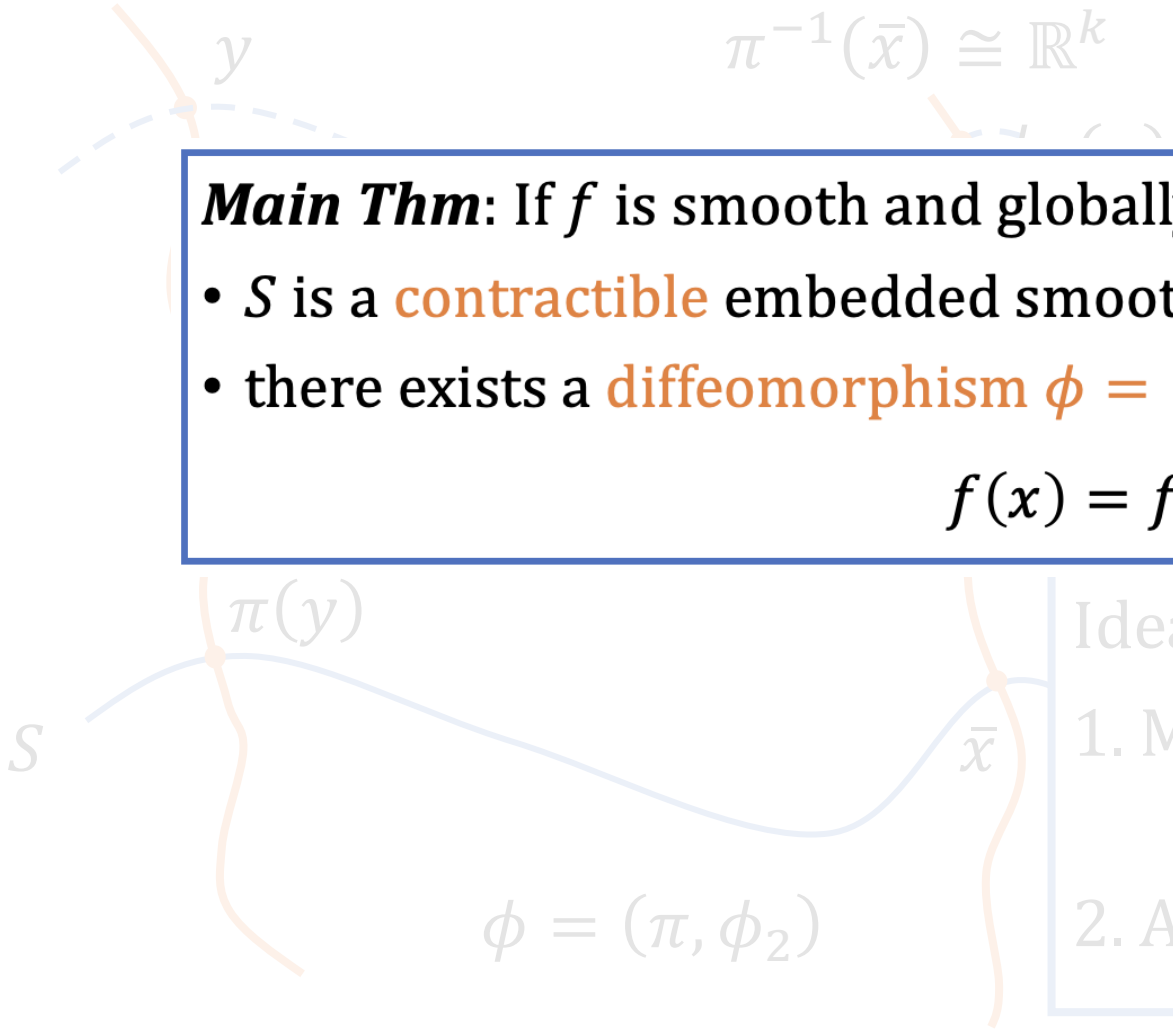

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Technical terminology:
$\pi: \mathbb{R}^n \to S$ is a trivial fiber bundle

Build your own crazy PŁ function

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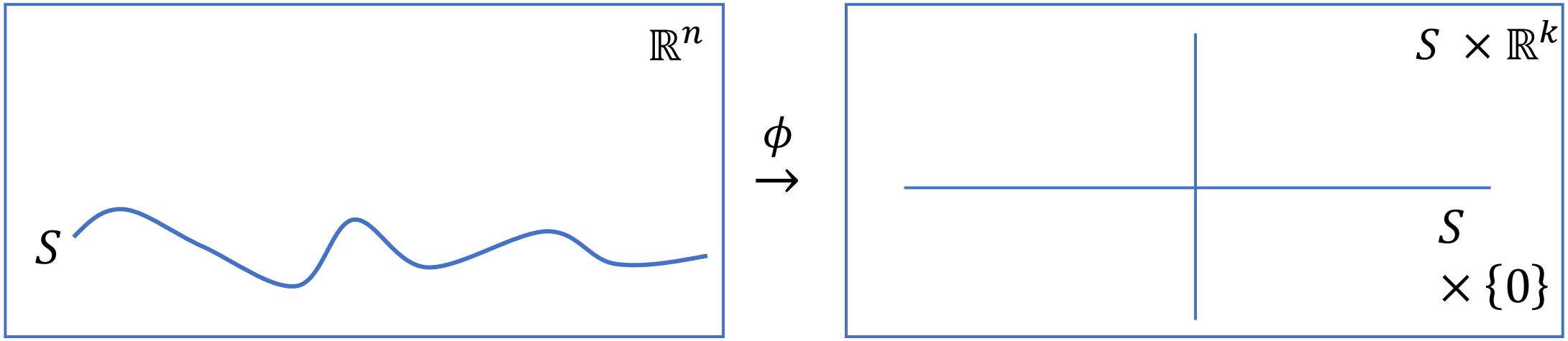

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Thm: If f is smooth, globally PŁ and non-constant on contractible M, then M is diffeomorphic to $\mathbb{R}^n$.

Punchlines + Questions

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Questions:

C^p regularity?

Make results quantitative?

Possible embeddings of $S \subset \mathbb{R}^n$?